Complicate Numbers and the Two-Hand-Clock, Vol. 1/2
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I just discover in feb-2018 that what I've called Gnomons, are known as Nexus Numbers or forward difference, backward difference etc... And the use of solving such problems (typically Integrals or Sum with an infinite number of Step) was called Umbral Calculus. But I hope will be clear after reading all the 2 Volumes, I was gone as deep as possible inside of each problem, discovering what was not yet fully investigated

I'm for so rewriting all my paper using the std. notation, where it is necessary / useful / possible.

From the previous public version of this work you will find several new chapters and new Tables, to prove my work is genuine and I hope, still interesting and new.

Reference to the "official" known math can be found at:

http://mathworld.wolfram.com/ForwardDifference.html

http://mathworld.wolfram.com/UmbralCalculus.html

Being free from what is known and what none, I was free to discover some new things seems not jet know.
My simple point of view will not be present in books that are considered as milestone for this field of Math like:

E.T. WHITTAKER and GIULIA ROBINSON: CALCULUS OF OBSERVATIONS

Abramowitz, M. and Stegun, I. A. (Eds.). "Differences." §25.1 in Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing. New York: Dover, pp. 877-878, 1972.

Beyer, W. H. CRC Standard Mathematical Tables, 28th ed. Boca Raton, FL: CRC Press, pp. 429-515, 1987.

Boole, G. and Moulton, J. F. A Treatise on the Calculus of Finite Differences, 2nd rev. ed. New York: Dover, 1960.

Conway, J. H. and Guy, R. K. "Newton's Useful Little Formula." In The Book of Numbers. New York: Springer-Verlag, pp. 81-83, 1996.

Fornberg, B. "Calculation of Weights in Finite Difference Formulas." SIAM Rev. 40, 685-691, 1998.

Iyanaga, S. and Kawada, Y. (Eds.). "Interpolation." Appendix A, Table 21 in Encyclopedic Dictionary of Mathematics. Cambridge, MA: MIT Press, pp. 1482-1483, 1980.

Jordan, C. Calculus of Finite Differences, 3rd ed. New York: Chelsea, 1965.

Levy, H. and Lessman, F. Finite Difference Equations. New York: Dover, 1992.

Milne-Thomson, L. M. The Calculus of Finite Differences. London: Macmillan, 1951.

Richardson, C. H. An Introduction to the Calculus of Finite Differences. New York: Van Nostrand, 1954.

Spiegel, M. Calculus of Finite Differences and Differential Equations. New York: McGraw-Hill, 1971.

Stirling, J. Methodus differentialis, sive tractatus de summation et interpolation serierum infinitarium. London, 1730. English translation by Holliday, J.

The Differential Method: A Treatise of the Summation and Interpolation of Infinite Series. 1749.

Tweddle, C. James Stirling: A Sketch of His Life and Works Along with his Scientific Correspondence. Oxford, England: Oxford University Press, pp. 30-45, 1922.

Weisstein, E. W. "Books about Finite Difference Equations. "http://www.ericweisstein.com/encyclopedia

Zwillinger, D. (Ed.). "Difference Equations" and "Numerical Differentiation." §3.9 and 8.3.2 in CRC Standard Mathematical Tables and Formulae. Boca Raton, FL: CRC Press, pp. 228-235 and 705-705, 1995.

...etc...

Note for beginners:
- To refresh the knowledge on Sum's Rules, you can read Appendix 1

You can find animated Gif, upgrade and other info at my webpage:

http://shoppc.maruelli.com/two-hand-clock.htm

http://shoppc.maruelli.com/two-hand-clock/MARUELLI-TWO-HAND-CLOCK-ANI.gif

To Send your comments to the Author use the email: robotec2@netsurf.it

Abstract:

This paper represent my investigation, from 2008, in Sums, Power's properties and related problems.

The core of the work is a consequence of the Telescoping Sum Property, that allow us to square all the derivative of the functions $Y = X^n$, via a Sum of Rectangular Columns called Gnomons.
Year by years my math skill with (and without) this new toys rise, so each time I have to return to the beginning of the story to rewrite all.

I'll present here Numbers and Sums in a New Vest that will take us to Limits and Integrals in a simple way, similar to the known Riemann Integral one, that probably Fermat has discover too.

A new simple Two-Hand-Clock shows how this Additive Modular Algebra works, and gives the name to this paper: I've called this: Complicate Modulus Algebra. It involves $Complicate Numbers$ that can be connected to the Set theory concept of Ordinal Numbers.

The first goal was to have a powerful instrument to attack the Power problems like Fermat and Beal, that I finally relegated to the Vol.2 to be sure all the basic concept was fully clear (and errors free as possible).

I discover that the representation of Rational and Natural numbers just via Complicate Numbers, is reductive, since it is also possible to represent Irrationals, so once again, all the paper was rewritten.

As minor consequence of the discover of the Complicate Numbers is a simple algorithm to extract the n-th root from any Number P (also by hands). Some simple relation, like the one between x^n and $n!$ will be shown too.

Il the Volume, A, B, C are usually Integers. n is used to define the exponent of the $Y = X^n$ functions we consider in this volume, so when we talk of *derivative* we refer to: $Y' = nX^{n-1}$, and when we talk of "All the derivative" we mean: $Y', Y'', Y''' etc...$ in general till the "significant one" that is for us the last depending by X (so the linear one so the last it is non a Constant).

Since I'll refer all to the Cartesian Plane, instead of using the index i we will use X and I'll show how this simple trick will open a new door in the investigation of Power proprieties.
In the Appendix 1 I summarize All the known Sum properties, and some trick can be done using this Sum properties.

In the Vol.2 I'll investigate in some very difficult Number Theory problems involving Power of integers like Fermat's The Last and the Beal conjecture, and I'll show how to approach to Riemann's zeros using Ordinal Numbers.

The level of the presentation is for undergraduate students so I several time repeat the concepts also using pictures.

Stefano Maruelli

Montalto Dora, Noth West Italy - From 01-08-2008 so far

Chapt.1: Modular Algebra vs Complicate Modulus Algebra'

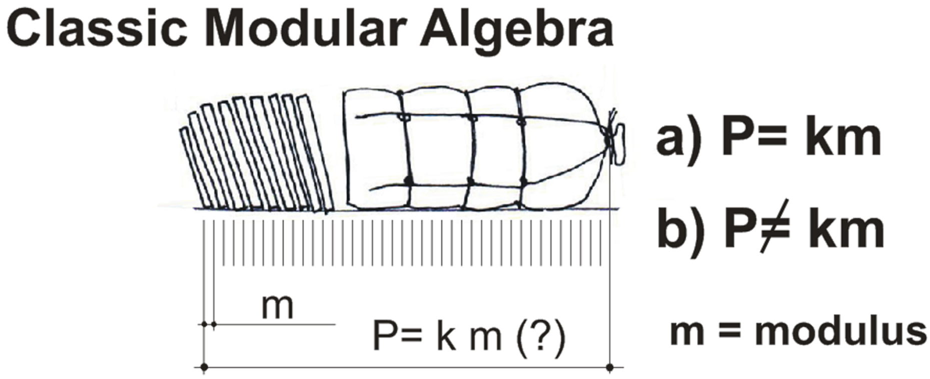
Introduction: Classic Modular Algebra

In Classic Modular Algebra we cut the salami in slices of same thickness, and we have just two case:

a) $P = K * m + 0$ or

b) $P = K * m + (Rest \neq 0)$,

and we don't care about the number of slice we cut, but just if we have or not a *Rest* and how Big it is. And then we make concerning on the Class of the Rest, and we talk of Congruences to solve our problems (etc.).



My question, born in a brainstorming session with my wife (wile pushing the car with my little new son) was: Is it possible to think to a different, more useful way to (always or in certain case) cut the Salami ?

The answer I found very quickly is YES: instead of cutting the Salami with same thickness slice, we cut it with Rising Slices following an useful Function I've called COMPLICATE MODULUS.

The result is that we have now in the hands a measurable collection of rising Integer Parts and (in case) a Rest. The advantages respect to the classic modular algebra are:

1- We Always have back exactly the number we put in, so a Weighted Zeros that told us How Many Cut we did, plus a Rest (in case).

It means that we can distinguish from the first, the second or the n.th slice (zero) we are talking of, and this will be very useful, and let (as the case we present here), intact the bijection also with the Real, Rational and/or Integer Numbers we have in the hands.

2- Once we fix the $n - th$ power of our interest we, at the same time, fix the COMPLICATE MODULUS, I call here M_n so the Class of Rest=0 shows us a Special Number of our interest is an n-th Power (of an Integer at the moment).

Day by day from that August 2008, new interesting aspect of this initial idea was discovered and it seems there is no end in new interesting properties and /or new problems / theorems.

Al starts from this simple table involves Newton's develop and what I later discover are called Nexus Numbers, that are all the recursive numbers we can obtain starting from plotting integers and it's powers. In the following example Squares and Cubes:

Integers	Power(s)	the Nexus Number(s)		
X	Y=X^2	Y'_i =X^2_(I+1)-X^2_I	Y''= Y'_(I+1)-Y'_I	Y'''= Y''_(I+1)-Y''_I
1	1	1	1	1
2	4	3	2	1
3	9	5	2	0
4	16	7	2	0
5	25	9	2	0
6	36	11	2	0
7	49	13	2	0
8	64	15	2	0
9	81	17	2	0
10	100	19	2	0

Integers	Power(s)	the Nexus Number(s)			
X	Y=X^3	Y'_i =X^3_(I+1)-X^3_I	Y''= Y'_(I+1)-Y'_I	Y'''= Y''_(I+1)-Y''_I	Y''''= Y'''_(I+1)-Y'''_I
1	1	1	1	1	1
2	8	7	6	5	4
3	27	19	12	6	1
4	64	37	18	6	0
5	125	61	24	6	0
6	216	91	30	6	0
7	343	127	36	6	0
8	512	169	42	6	0
9	729	217	48	6	0
10	1000	271	54	6	0

Etc....

Definition of the Complicate Modulus Algebra

All born from this simple rule:

Square of Integers as Sum of Odds

It's well known that a square of an Integer p, is equal to the sum of the first p Odds (for the Telescoping Sum Property):

$$a^2 = \sum_{i=1}^{a} (2i - 1)$$

The proof it's very easy developing the Sum we have:

$$\sum_{i=1}^{a} (2i-1) = a^2 - (a-1)^2 + (a-1)^2 - (a-2)^2 + (a-2)^2 - ... + 1 - 1 = a^2$$

So for example $a = 5$; $a^2 = 1 + 3 + 5 + 7 + 9 = 25$

This simply known rule suggest me a new branch of Modular Arithmetic and Set Theory, I've called COMPLICATE MODULUS ALGEBRA.

Complicate Modulus Algebra Idea

The above property suggest to me that this kind of division of the Squares can be taken as example of a new kind of Modular Algebra where instead of a Fixed Integer Divisor m, there is a *Function* that define the thickness of each following slice.

I'll present here the case where this function is a known continuous rising function, coming from the Telescoping Sum property for Power of Integers (but will also hold for Rational, and, at the limit, for the Reals) that produce each time larger and large slices.

So from the easy known rule for Squares ($n = 2$):

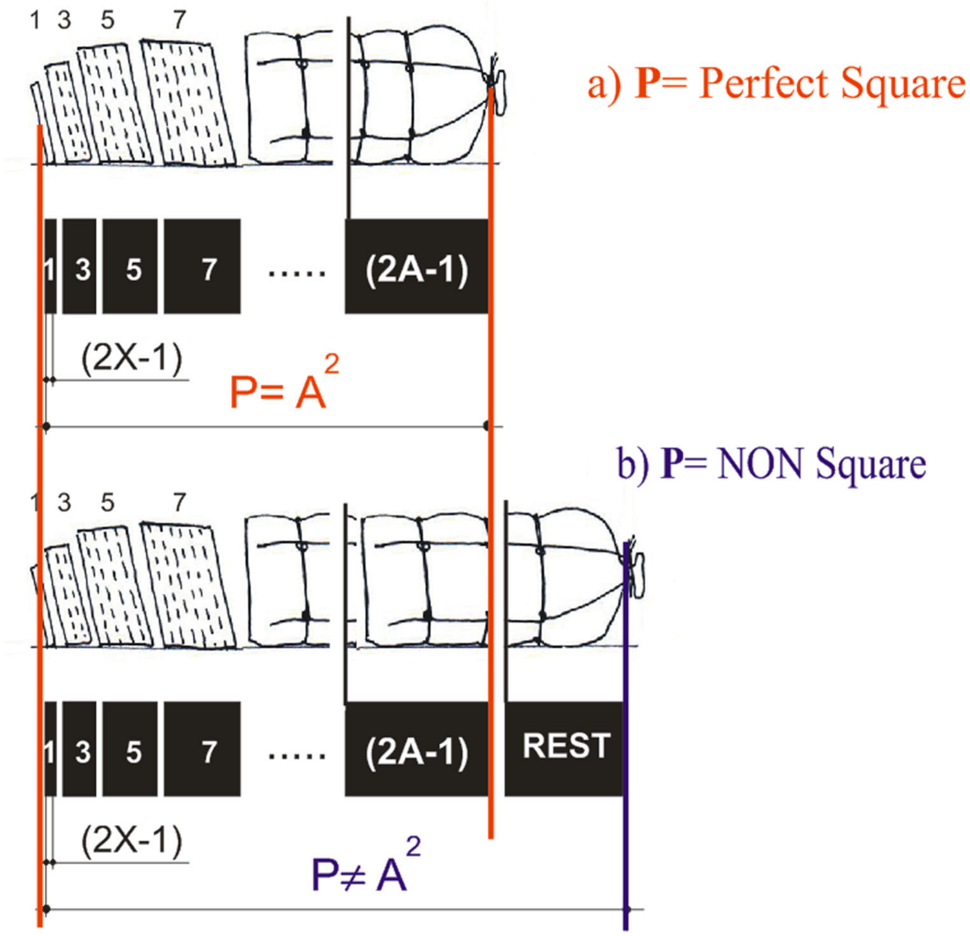
$$a^2 = \sum_{X=1}^{a} (2X - 1)$$

We can see the Complicate Modulus $M_n = M_2$ where the slice's thickness linearly rise following the function: $M_2 = (2X - 1)$

As example, in the following picture How to Cut a Salami of Length $P \in \mathbb{N}$ in the case we take the Square complicate Modulus, so $n = 2$; $M_2 = (2X - 1)$. To let things more clear we will call p also the Integer Root of the Generic Number P we are studying. Or: $p = \lfloor (P^{1/n}) \rfloor$

Complicate Modulus Algebra

Case n=2 → Complicate Modulus M2= (2X-1)



As we can see we can distinguish 2 case:

a) P is a perfect Square, so we have no Rest, or

b) P is not a perfect Square, than we have a Rest.

We can also see that for $n = 2$, only, due to the linearity of the first derivative, is possible to have the same result p^2 using another Complicate Modulus, I call $M_{2,X+1} = (2x + 1)$, but at the condition that:

- We shift the Lower limit from 1 to 0:

- We shift the Upper Limit FROM a to $p - 1$, thanks to the Sum properties we can write:

$$P = p^2 = \sum_{X=1}^{p} (2X - 1) = \sum_{X=0}^{p-1} (2X + 1)$$

More in general, this shifting Rule is true for any $m \in \mathbb{N}^+$ (in the next chapters the general rule):

$$p^2 = \sum_{X=1}^{p} (2X - 1) = \sum_{X=1-m}^{p-m} (2X - 1 + 2m) = \sum_{X=1-m}^{p-m} (2(X + m) - 1)$$

So can now generalize the formula for our COMPLICATE MODULUS using the General Formula for n-th Power written trough the Telescoping Sum:

$$p^n = \sum_{X=1}^{p} (X^n - (X - 1)^n)$$

Where I've called the function defining the Terms of the Sum (coming from the Binomial Develop): M_n **the COMPLICATE MODULUS**

$$M_n = (X^n - (X - 1)^n)$$

Now we can generalize to All Numbers, representing (for example) the Natural, called P, Modulus M_n, fixing n as we need or prefer, as:

$P = pM_n + Rest$

having a $Rest = 0$ in case $P = p^n$ and $Rest \neq 0$ in ALL the other case.

With this new Algebra we have back more information than the Classic Modular one since this preserve the bijection between integers and the Complicate Modulus Numbers. And I'll show how we can go over in $\mathbb{Q}$ and $\mathbb{R}$ under certain conditions.

Note: Complicate Numbers vs Complex Number

There is a big difference between a Real Complicate Numbers that is an element of $\mathbb{R}$, so a point on a one-dimensional number line, that can be expressed under my conditions on a two-dimensional real plane and a Complex Number that is an element of $\mathbb{C}$ and that can be represented just on a two-dimensional complex plane, since the Rest, vice versa from the Imaginary Part of the Complex, strictly depends on what we choose as Integer Root.

It means that Rest and Integer Root are connected with a sort of Gear, while this doesn't, usually, happen in a Complex Number, except if we fix a Relation between the Imaginary part, and the Real one.

Complicate Numbers as Ordinal Number

In late '900 the ZFC set theory jumps on the math scenario adding several new concepts on of those is the Ordinal Number.

I'll introduce here the Ordinal Number concept without saying more on that, but we use them in the Vol.2 to prove Fermat the Last. At the moment, it's just necessary to know that:

Here I call the Ordinal Number $M_n = (X^n - (X-1)^n)$ the **Complicate Modulus** since we can use it to represent any Natural Number as Sum of its Greatest Integer n-th Root, plus an Integer Rest. In case we are talking of Squares we can (usually) write:

P= (Integer n-th Root)^n + *Rest* or using the proper Math Floor Brace $\lfloor ... \rfloor$ Symbols (ex. $n = 2$):

$$P = \left\lfloor \sqrt{P} \right\rfloor^2 + \left(P - \left\lfloor \sqrt{P} \right\rfloor^2 \right) = \lfloor \sqrt{P} \rfloor M_n + Rest$$

And more in general:

$$P = pM_n + (P - p^n)$$

So we have a bijection from $\mathbb{N} \rightarrow \mathbb{M}_{\ltimes}$, where $\mathbb{M}_{\ltimes}$ is the Set of this Complicate Numbers, base n we have chosen, as shown for example in the next Table:

Table 1: Naturals rewritten via $M_2 = 2X - 1$ so as Square plus Rest

X	M_2	Rest		
1	1	0	$= 1^2 + 0$	$= 1M_2 + 0$
2	1	1	$= 1^2 + 1$	$= 1M_2 + 1$
3	1	2	$= 1^2 + 2$	$= 1M_2 + 2$
4	2	0	$= 2^2 + 0$	$= 2M_2 + 0$
5	2	1	$= 2^2 + 1$	$= 2M_2 + 1$
6	2	2	$= 2^2 + 2$	$= 2M_2 + 2$
7	2	3	$= 2^2 + 3$	$= 2M_2 + 3$
8	2	4	$= 2^2 + 4$	$= 2M_2 + 4$
9	3	0	$= 3^2 + 0$	$= 3M_2 + 0$
10	3	1	$= 3^2 + 1$	$= 3M_2 + 1$
11	3	2	$= 3^2 + 2$	$= 3M_2 + 2$
12	...	...	...	...

Here I'll call the $i - esim$ Square Gnomon the $i - th$ value it will assume M_2, so:

$$M_{2,i} = (2X - 1)_{X=i} = 2i - 1$$

So it's similar to the well known Modular Arithmetic, where Squares just, are our Zeros, in fact:

If and only If $P \in \mathbb{N}^+; P = p^2$ then we have $Rest = 0$, As I'll present in the Chapt.4, the big difference with the Old Modular form, is that now we have a New Clock that Shows us the Right Hour All Day Round, and moreover, it shows us exactly when our Number P is a certain n-th Power, we decide, of an Integer p, or not. So, for example, there is no longer confusion like between 12 and 24 that are indistinguishable modulo 2, and numbers (squares for examples) $1, 4, 9, 16 ... x^2$, will be always in evidence to the observer, for example, still if randomly written in a list of integers.

Another example of the bijection in case $n = 3$

Table 2: Naturals rewritten via $M_3 = 3X^2 - 3X + 1$ so as Cube plus Rest

X	M_3	Rest		
1	1	0	$= 1^3 + 0$	$= 1M_3 + 0$
2	1	1	$= 1^3 + 1$	$= 1M_3 + 1$
3	1	2	$= 1^3 + 2$	$= 1M_3 + 2$
4	2	0	$= 1^3 + 3$	$= 1M_3 + 3$
5	2	1	$= 1^3 + 4$	$= 1M_3 + 4$
6	2	2	$= 1^3 + 5$	$= 1M_3 + 5$
7	2	3	$= 1^3 + 6$	$= 1M_3 + 6$
8	2	4	$= 2^3 + 0$	$= 2M_3 + 0$
9	3	0	$= 2^3 + 1$	$= 2M_3 + 1$
10	3	1	$= 2^3 + 2$	$= 2M_3 + 2$
11	3	2	$= 2^3 + 3$	$= 2M_3 + 3$
12	3	3	$= 2^3 + 4$	$= 2M_3 + 4$
13	...	...	...	...

So more in general, but not jet in the Most General Case, we present a Complicate Number as:

$$P \in \mathbb{R} : P = \left(\lfloor P^{(1/n)} \rfloor \right)^n + Rest = \sum_{X=1}^{\lfloor P^{(1/n)} \rfloor} M_n + Rest$$

Or in the new Complicate Modulus notation as:

$$P \in \mathbb{N} \text{ or } P \in \mathbb{Q} \text{ or } P \in \mathbb{R} : P = pM_n + Rest$$

As we will see in the next pages there is a more general series of Definitions for a Complicate Number.

Definition of a Complicate Numbers:

After presenting the Complicate Modulus Numbers in the "soft, informal" mode, is time for a General Definitions:

A Complicate Modulus Number P, (from now: Complicate Numbers, just) is a number that can be expressed in the form:

$$P = p^n + Rest$$

Where more in general than what presented in the previous pages:

$$(P,\ Rest,\ p) \in \mathbb{R}$$

And of course p is often defined as:

$$p = \lfloor (P^{1/n}) \rfloor$$

where for the non expert $\lfloor, \rfloor$ is the Floor operator that return the $IntegerPartOf$ what in the middle of this special Bracket. Often, since it is the most useful representation (so a special case of Complicate Modulus Number representation, we will see later) but of course p can also be any integer under the right conditions.
Definition of what a Rest is:

The $Rest$ is considered an often (not always) Positive Number, and trivially defined as:

$$\textbf{Rest} = \textbf{P - p}\ M_n$$

Where using a new notation that helps to understood we are talking of a Complicate Number, and considering that the Rest can be taken with a sign we decide, we can write:

$$P = pM_n + / - Rest$$

As we will see in the next pages in CMA the Rest can assume a value that is Bigger than the Power we are analyzing, so it will require some care working with it.

So under a series of conditions I'll show better later, we can distinguish several Class of Complicate Numbers:

1) Integer Complicate Numbers:

Is the most used and happen when:
$(P\ ,\ Rest\ ,\ p) \in \mathbb{N}$;

Another possible, sometimes useful, Class is:

2) Rational Complicate Numbers:

$(P$ and/or $Rest$ and/or $p) \in \mathbb{Q}$

3) **Real Complicate Numbers:**

Is the widest class, and can be divided in:

3a) Real Complicate Integer Numbers:

$(P\ ,\ Rest) \in \mathbb{R}$; $p \in \mathbb{N}$;

3b) Real Complicate Rational Numbers:

$(P\ ,\ Rest) \in \mathbb{R}$; $p \in \mathbb{Q}$;

3c) Real Complicate Number:

$(P\ ,\ Rest\ ,\ p) \in \mathbb{R}$;

And in each of the previous Class we can define other 2 Sub-Classes depending on the Rest:

a) **Reduced Complicate Modulus Number**, is defined as above:

$$P \in \mathbb{R} | P = pM_n + Rest = \left(\lfloor P^{(1/n)} \rfloor\right)^n + Rest = \sum_{x=1}^{\lfloor P^{(1/n)} \rfloor} M_n + Rest$$

Where (again) $p = \lfloor (P^{1/n}) \rfloor$

So p is (usually) the Maximum Integer n-th Root of P (as defined by the Floor Brace operator), and the $Rest$ is the Minimum Rest we can have once we choose the desired n and the Maximum Integer n-th Root $p = p_{max}$ (in case we are working with what we define a Reduced Complicate Modulus Number)

The $Rest$ for a Reduced Complicate Modulus Number, here representing an integer Integer P written as $P = p^n + Rest$ is for so bounded by the Rules:

$$Rest_{min} = 0$$

$$Rest_{Max}(P) = M_n|_{x=p_{max}} = [(p_{max})^n - (p_{max} - 1)^n]$$

So in words talking of Reduced Complicate Modulus Number, the $Rest_{Max}$, for so also the generic *Rest* of such Reduced Complicate Modulus Number, are both strictly littlest than the Next Power $(p+1)^n$ and of course also of the Next Gnomon $M_n|_{x=p+1}$ value.

I've shown the case we choose an Integer Complicate Modulus, because it require more concerning in the other case we will see later (there is also a Non Integer Complicate Modulus Algebra !)

We can also ask who is the Better Optimized choice for p and n to let the *Rest* be the *Absolute Minimum Rest*, especially in the case $P \in \mathbb{R}$, opens another branch of math (I call it another Black Hole... since very wide, deep, and hard to be studied)

Similar cases are studied in Classic Math in Diophantine approximation and are known as Roth's theorems (etc.).

From Wikipedia:

In mathematics, Roth's theorem is a fundamental result in Diophantine approximation to algebraic numbers. It is of a qualitative type, stating that a given Algebraic Number α may not have too many rational number approximations, that are -very good-. Over half a century, the meaning of very good here was refined by a number of mathematicians, starting with Joseph Liouville in 1844 and continuing with work of Axel Thue (1909), Carl Ludwig Siegel (1921), Freeman Dyson (1947), and Klaus Roth (1955).

Where I remember: *an Algebraic Number is any complex number that is a root of a non-zero polynomial in one variable with rational coefficients (or equivalently -by clearing denominators- with integer coefficients). All integers and rational numbers are algebraic, as are all roots of integers. The same is not true for all real and complex numbers because they also include transcendental numbers such as π and e. Almost all real and complex numbers are transcendental.*

b) Non Reduced Complicate Modulus Numbers:

I hope is now clear we can write a number P as:

$$P \in \mathbb{R} : P = pM_n + Rest$$

Where we have two case (so we call the root a instead of p to distinguish it better):

1) $a < \lfloor P^{(1/n)} \rfloor$, so the *Rest* is NOT the *MinimumRest*, so it is bounded in function of how big is the a we have chosen.

2) M_n itself can be a NON Reduced Complicate Modulus, so, more in general it can be:

$$M(nr)_n = [r(X^n) - r(X-1)^n]; (X, M_n) \in \mathbb{Q}; (r, Rest) \in \mathbb{R}$$

Where: **r** is a constant, for example, useful to better shows

$$\text{Perfect Powers } P = r * a^n + 0$$

In this Volume where I wanna show the basic of this Algebra, usually **r** $= 1$, but more often while working on real problems is $r \in \mathbb{N}$ or $r \in \mathbb{Q}$).

I'll show more ahead in this Volume, and in the next Vol.2, that can be also $r \in \mathbb{R}$ under certain conditions (for Fermat Last Theorem For example and in case we work with Irrational Numbers), for example depending by Known Irrational Factors.

Where **Rest:** is the Rest, so what cannot be, or is not written trough the n-th power of a^n, or its multiple $r * a^n$.

In this class the Rest has larger bound is: $Rest \leq P$

The trivial case: $Rest = P$ is, of course, useless.

And where **a** is for so bounded by:

$$0 < a < \lfloor (P^{1/n}) \rfloor \ , \ a \in \mathbb{N}$$

The most General Definition for $P \in \mathbb{Q}$ and for $P \in \mathbb{R}$ will follows after we have made a little tour on what we can do with this numbers.

Definition of $\mathbb{M}_{\mathbb{R}}$, the Set of the Real Complicate Numbers:

We can now give a name to the biggest Set of the (generic) Complicate Modulus Numbers that, as already shown, will depend on the Complicate Modulus M_n or $M(nr)_n$we chose as Base of our bijection. We will call:

$\mathbb{M}_n =$ the Set of Complicate Modulus Numbers base n

After the definitions I've already given is clear: $\mathbb{M}_{\mathbb{R}} = \mathbb{R}$

We, mostly, investigate in some Sub Set of $\mathbb{M}_{\mathbb{R}}$, like $\mathbb{N}^+$ or $\mathbb{Q}^+$

As told We can also create an Imaginary Complicate Number (etc...), but this will not investigated here.

Some concerning on the Complicate Numbers:

- The fact that we can distinguish from Reduced and Non Reduced Complicate Numbers will probably give some property I not jet investigate here, but I suspect it will be connected to the Proof of the Beal Conjecture will follow in the next chapters.

- In Classic Modular Algebra the *Class* of *Rest* are an important part of that theory, while here I've not jet investigated if some property of the *Rest* will be useful and where, still if I'm quite sure "it will be somewhere" once well studied.

The most general **Complex Complicate Modulus Number** definition follows from the above definitions and the classic definition of a Complex Number. I've not jet investigated this field.

I'll present hereafter how a Complicate Modulus Number behave during classic known operations.

After that I'll present a nice easy way to show on a Cartesian plane who is a Reduced Complicate Modulus Number (from here a Complicate Number) and its Gnomons.

Fundamental Operations with Complicate Numbers:

For "classic" computation this numbers are not useful, in fact:

Sum of two Complicate Numbers:

Unfortunately the Sum for example: $A = 5 = 2M_2 + 1$ and $B = 11 = 3M_2 + 2$ cannot be done so easy as to Sum the Integers $A + B = 5 + 11 = 16$, because the Clock change Number of Division Each Turn, so the Rest and the Integer Root, cannot be simply summed one by one, but we need to return each time to the Original Integer Numbers we are considering and then make the computation:

- We have to re-transform the Integer Root of A and the Integer Root of B in the correspondent Powers (So in Natural Numbers):

$2M_2 + 3M_2 = 4 + 9 = 13$

Than we can Sum the Rests:

$(1) + (2) = 3$

Than we have the result:

$A + B = (13) + (3) = 16$

Than we can return to our M_2 base:

$A + B = (2M_2 + 1) + (3M_2 + 2) = (13) + (3) = 16 = 4M_2 + 0$

Difference of two Complicate Numbers:

The difference follows in the same way, so the Integer Root Part has to be transformed again in a Natural Number:
Having for example: $A = 5 = 2M_2 + 1$; $B = 11 = 3M_2 + 2$ we have:

$B - A = 11 - 5 = 6$ so to perform the subtraction: $(3M_2 + 2) - (2M_2 + 1)$

- We have to re-transform the Root into the Natural Numbers, then we can subtract:

$3M_2 - 2M_2 = 9 - 4 = 5$

Than we can Subtract the two Rests (with sign, in case is negative):

$(2) - (1) = 1$

Than we have the result:

$B - A = (5) + (1) = 6$

And then we finally transform 6 in a Complicate number base Squares, and we have our result: $(3M_2 + 2) - (2M_2 + 1) = (2M_2 + 2)$

Interesting case in the Sum of Complicate Numbers:

We know that there can exist some triplets A,B,C for what:

$A^2 + B^2 = C^2$

so for certain (A, B) happens that $A^2 + B^2 \in \mathbb{N}^{+n}$

But this is an exception to the general Rule for Summation of Complicate Numbers (so also for Powers of Integers) that state that:

The Sum of two or more Complicate Numbers, having both Rest equal to Zero is not, always, again a Complicate Numbers having Rest equal to Zero.

Fermat understood (and claim to have a proof) that it's true that: $A^n + B^n <> C^n$ if $n > 2$

I'll prove this result in the next Vol.2 using the property we will learn in this Volume (still if till now all this can seems an useless complication, just).

We will see in the next chapters, when we present our Complicate Clock that there will be another way to Sum the Complicate Numbers, involving a New Clock and the angles of the Hands.

How to Divide a Complicate Numbers:

Having for example: $B = 10 = 3M_2 + 1$; $A = 2 = 1M_2 + 1$ we have:

$B/A = 10/2 = 5$ so: $(3M_2 + 1)/(1M_2 + 1) = (2M_2 + 1) = 5$

- Where again we have to return A and B in $\mathbb{N}$, make the division, than return the result to the new Complicate Number.

So there is no sense to do this operation.

How to multiply the Complicate Numbers: Rule is the same we use for Binomial Product, or Complex Numbers:

Having for example: $A = 5 = 2M_2 + 1$; $B = 11 = 3M_2 + 2$ we have:

$A \cdot B = 5 \cdot 11 = 55$ so: $(2M_2 + 1) \cdot (3M_2 + 2) = 55$

- For the Integer Root Parts we can make the direct multiplication, but we have to return in $\mathbb{N}$:

$\Pi_{IntRoot} = (2M_2) \cdot (3M_2) = 6M_2 = 36$

and Both the Rests we can make the direct multiplication:

$\Pi_{Rest} = (1) \cdot (2) = 2$

- While We have to re-transform in the Natural Numbers for the Mixed Products:

$\Pi_{Mix1} = (2M_2) \cdot (2) = 4 \cdot (2) = 8$

$\Pi_{Mix2} = (3M_2) \cdot (1) = 9 \cdot (1) = 9$

$Result = \Pi_{IntRoot} + \Pi_{Rest} + \Pi_{Mix1} + \Pi_{Mix2} = 36 + 2 + 8 + 9 = 55$

$$A \cdot B = (2M_2 + 1) \cdot (3M_2 + 2) = 7M_2 + 6$$

The Product seems again an useless work, but suggest us that there are 3 types of Complicate Numbers:

- Pure Powers: $p = A^n = aM_n$

- General Complicate Numbers

$p = aM_n + Rest$

That are able to cover $\mathbb{N}$, but nobody can deny us to use also:

- Multiple of Pure powers: $m \cdot A^n = m \cdot aM_n$

This will becomes useful to investigate on Fermat's Last Theorem that will state that there cannot be a solution for $n > 2$ in $\mathbb{N}^n$ to:

$$C^n = A^n + B^n = \sum_{1}^{A} M_n + \sum_{1}^{B} M_n$$

So, thanks to the Sum properties, it can be written as this symmetric formulation:

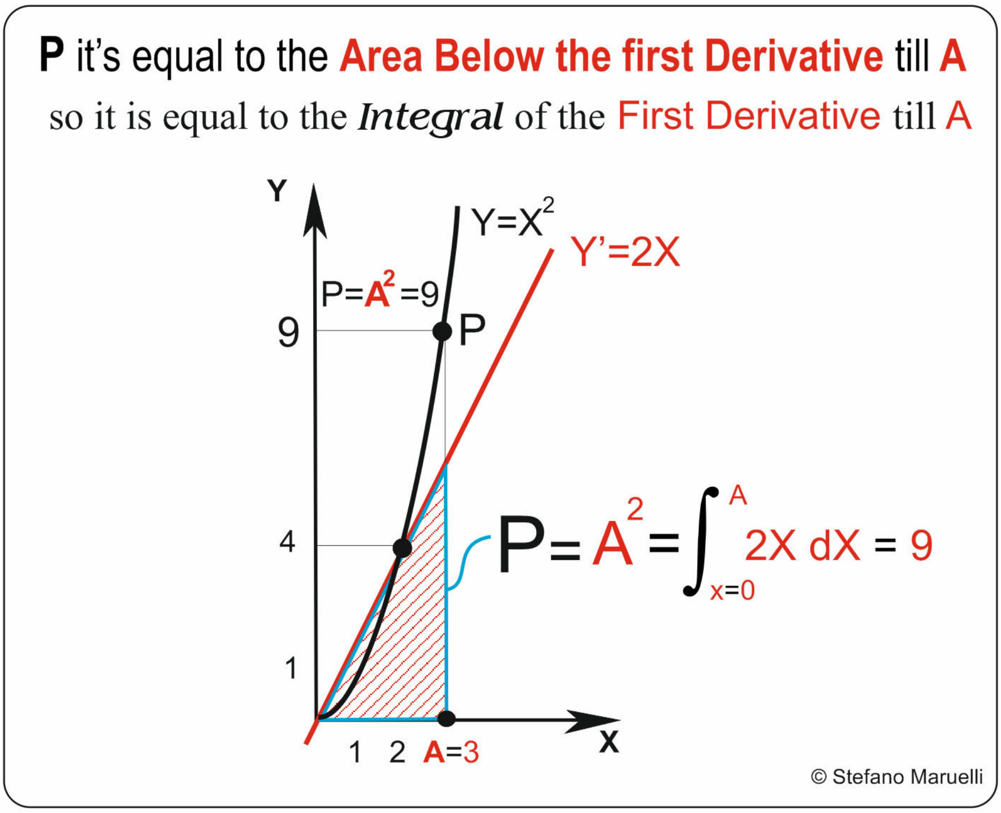
$C^n = 2A^n + \Delta$ and /or

$C^n = 2B^n - \Delta$ with $Delta = \sum_{A+1}^{B} M_n$

Where is clear we need to investigate the curve $y = 2x^n$ via $a[2M_n]$ Complicate Modulus. (See Vol.2: Fermat the Last Proof)

Chapt.2: Complicate Modulus Algebra on the Cartesian Plane:

Remembering what known from Classic Calculus, so that the value of the abscissa on an integrable function (here a parabola) is equal to the area of it's derivative till it's Abscissa:



As seen in the most simple example for Square, the Complicate Modulus $M_2 = (2x-1)$ produce odd numbers $(1, 3, 5, 7..., 2i-1)$ that can be represented on a Cartesian plane in the form of **Rectangular Areas** called from now **Gnomons** defined (till a more general definition will be given in the next chapters) by:

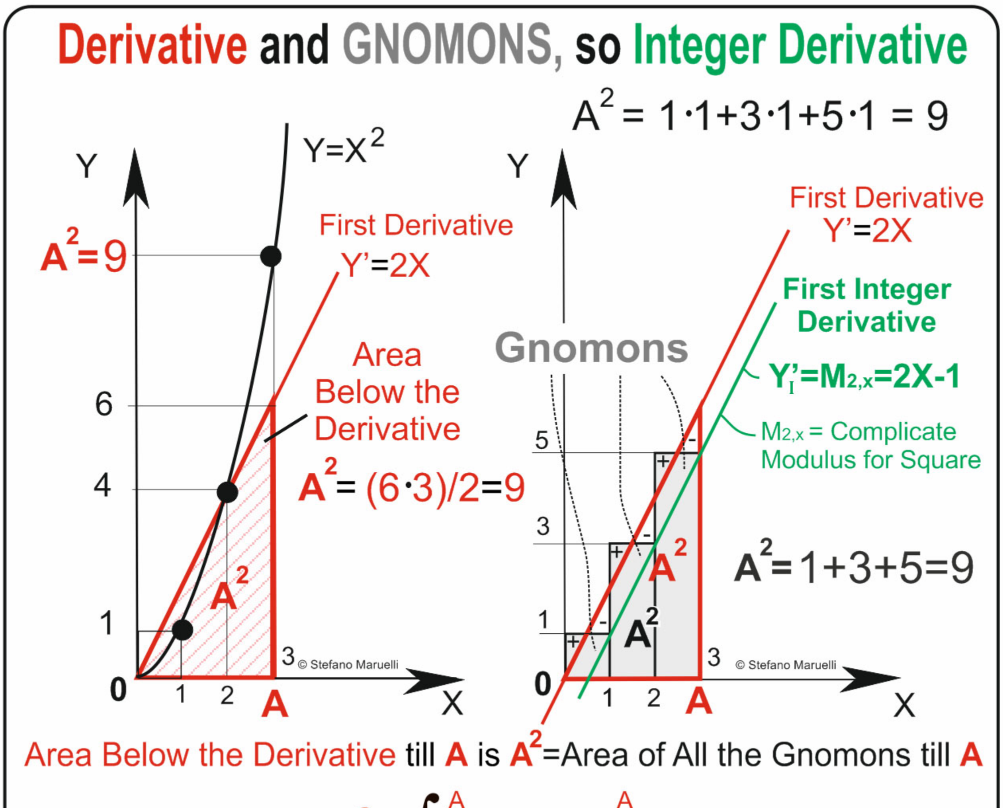
$$\textbf{Base}{=}1 \text{ (fixed value)} : \textbf{Height}{=}\ M_2 = (2X-1)$$

Where the i-th Gnomon's height is $M_{2,i} = (2i-1)$.

To well represent the Gnomons on the Cartesian plane, and to show it is connected to the area of all the derivative of $Y = X^n$ I need to change the Label's Index, from i to x since i becomes, as told, the i-esim Gnomon we are talking of :

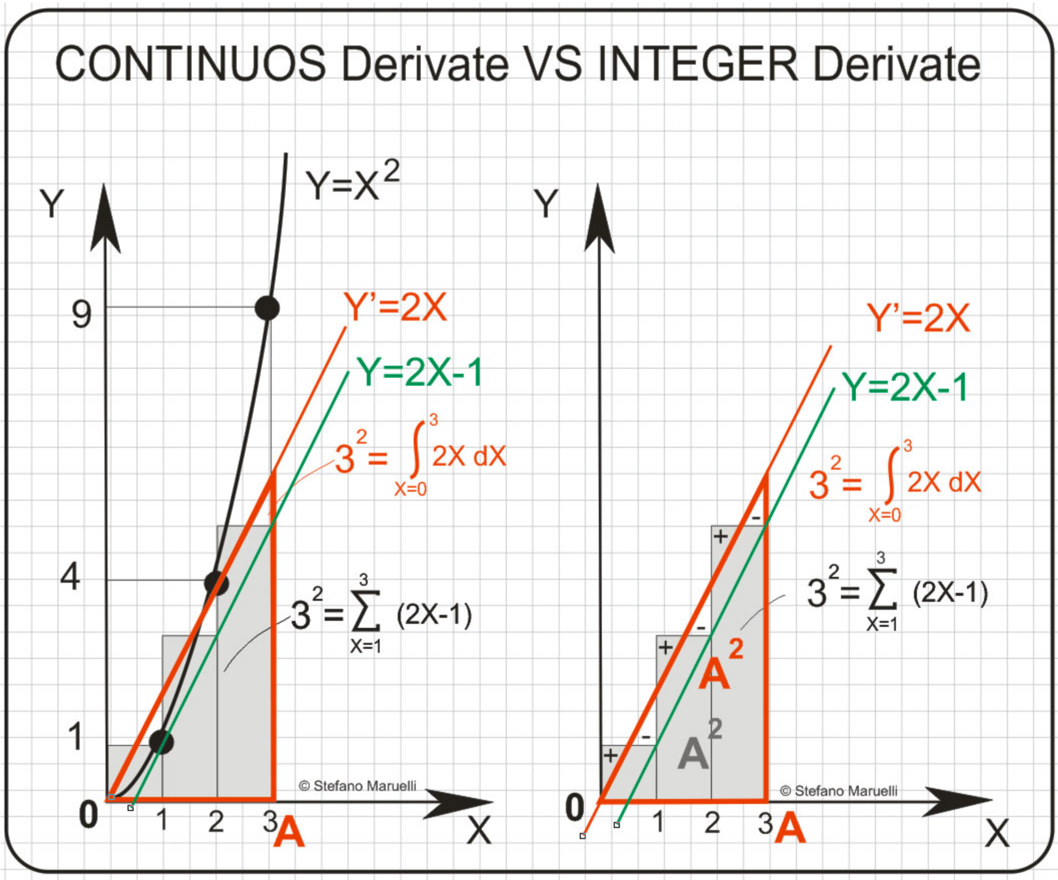
$$A^2 = \sum_{X=1}^{A} (2X-1)$$

I'll also use the uppercase A from now on, since this work started in this way in 2008, from the Fermat's A,B,C letters.
This Columns are the key of all my work. Here on the graph you can see how the Gnomons square the Linear First derivative $Y' = 2X$:



$$A^2 = \int_{x=0}^{A} 2x\,dx = \sum_{X=1}^{A} (2X-1)$$

In the following picture it is better shown the relation at the base of this formulation between: Sum of Gnomons and the Integral of the first derivative.



We know by the Integration method (or by the classic Triangular formula $b * h/2$) that the Triangular Area (Red Border) below the First derivative till A, is A^2, but we can square this Area also using the Rectangular columns called Gnomons (here in Grey).

This property can be extended to All Powers of Integers and to all the following (non flat) derivative.

We need to define few parameters on the picture to let be more clear what happens:

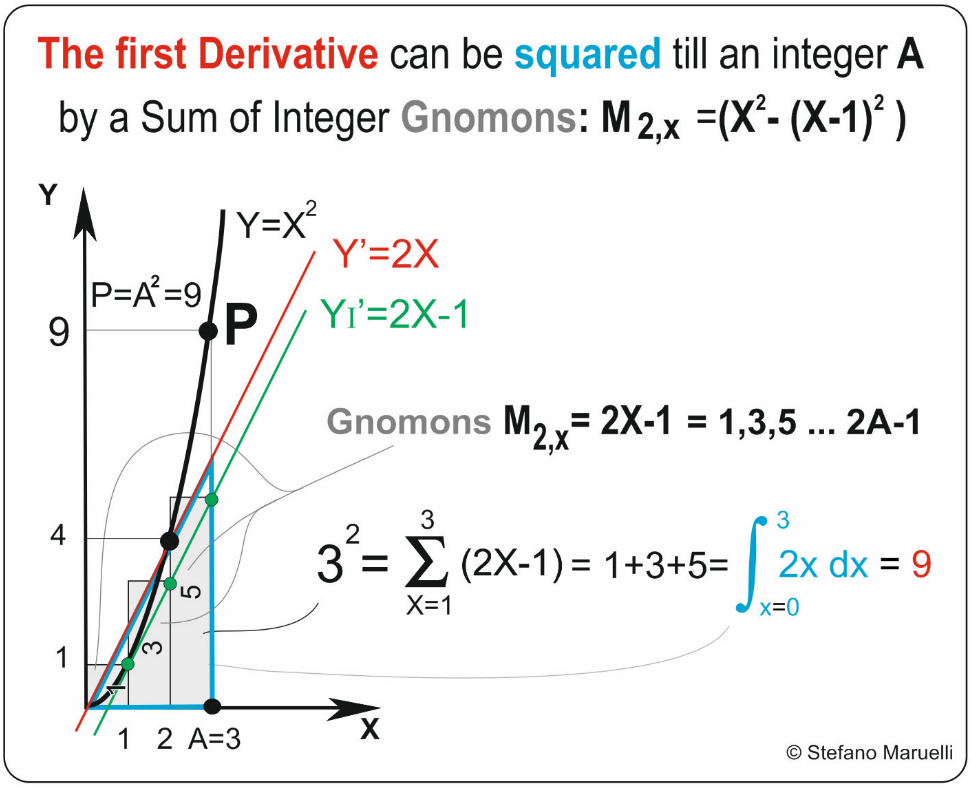


Figure 1: Gnomons squaring the First Derivative, and the First Integer Derivative

The telescoping Sum property has a Geometric reason: Gnomons square the First Derivative, thanks to the property Missing Area = Exceeding one
- The function $Y = X^n$, in the case $n = 2$ has a linear derivative: $y' = 2x$

- So it's possible to use the Gnomons (in Grey) to Square it,

At the condition that for each column of base $x_i - x_{i-1} \in \mathbb{N}^+$

- the Exceeding yellow Area A^+ between the flat roof of the Gnomon and the first derivative,

- is exactly equal to the Missing yellow one A^-.

For Square Powers of Integers, each Gnomon has:

- an unitary $Base = 1$ (we can see later the base can be different, under certain conditions, without loosing this property)

- an Height that rises of a linear value that lies on what I'll call:

Linear Integer derivative: $y_i' = 2x - 1$, so $1, 3, 5 ... (2xi - 1)$

But we have to go deeper now to investigate all the properties of this new subject.

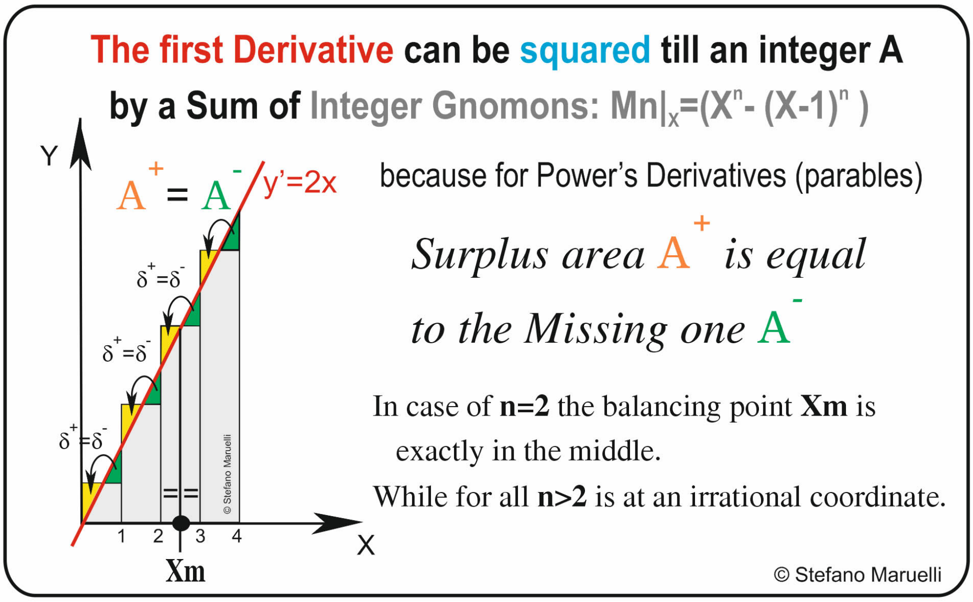


Figure 2: Gnomons squaring the First Derivative because Missing Area = Exceeding one

For $n = 2$ the derivative is linear, so it's clear what happens:

- it cuts the Roof of the rectangular Gnomons exactly in the middle,

so we have that for each Gnomon: $r = q$ and $Yr = Yq$

So the Exceeding area A^+ is exactly equal to the missing Area A^-, not just in value, but also in shape (triangular), so they have both the same Base, here $1/2$ and the same Height, here equal to $Yr = Yq = 1$.

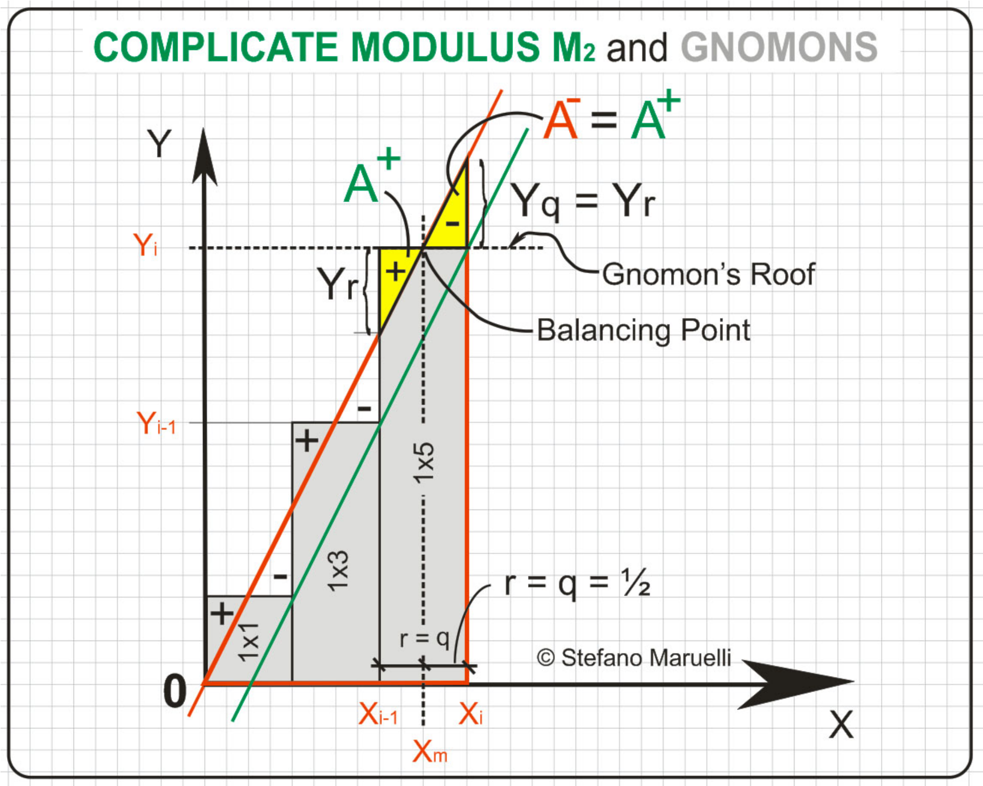
It's also clear that the Gnomon's Roof given by the $2x_i - 1$ formula is always an integer Value, for each Integer x_i,

Calling: **Balancing Point** the intersection between the First derivative and the Gnomon's Height. It has coordinate: (xm_i, y_i) where:

$$X_{m,i} = \frac{X_{i-1} + (X_i - X_{i-1}}{2} \implies X_{m,i} \in \mathbb{Q}$$

The fact the the Medium Point Xm_i is equal to the Balancing Point BP is due to the Linear First derivative.

This is no longer true for n>2.



Linear derivative and Pythagorean Triplets:

For n=2 just, if we define two integer abscissa $x_1 = A$, $x_2 = B$, there will exist an infinite number of integer $x_3 = C$ for what:

$$A^2 + B^2 = C^2$$

Pythagorean Triplets are possible because for n=2 the First derivative is Linear, so $X_m = BP$, the Second derivative is constant equal to 2, then moving right of any integer δx the area bellow the first derivative always grow of such constant value, and this assure that any new area is a Trapezium that itself respect the property $X_m = BP$

I stop here the concerning regarding this property since it will be presented in the Vol.2 when I'll present several reason why Fermat the Last theorem can be proved right. I just add here a trivial concerning:

Since the equality holds for those triplets, it will also holds the equation, with $a \in \mathbb{R}$:

$$aA^2 + aB^2 = aC^2$$

This will lead to known concerning about the $p*q$ factorization problem we will not discuss now.

Calling $\mathbb{P}^{\ltimes}$ the Set of the n-th Powers of Integers, is In General true that with A, B, n Integers:

$$A^n + B^n \notin \mathbb{P}^{\ltimes}$$

except for $n = 2$ where we know there exist an infinite number of Pythagorean Triplets.

Due to the Telescoping Sum Property I'll show in the next pages that we can use the same squaring process for higher n, so also when the derivative is a curve. Again we can square the derivative with Gnomons because for any $n \leq 2$, and for any Following derivative of the curve of the type $Y = aX^n$, the Exceeding area A^+ will equate the Missing A^- one.

This will happen still if the Exceeding / Missing Areas are no longer triangles, so they have not just different Bases and Heights, but also different shapes since the Left Exceeding Areas has a Concave Upper Border, while the right Missing one has a Convex Lower Border (see picture in the next chapter). The telescoping Sum Property, for example will not holds true for other curves, Hyperbola and Ellipsis, for example (as I'll show at the end of this Vol.1).

Chapt.3 Generalization: Powers as Sum of Gnomons:

Theorem1: from $n = 2$ it's possible to square all the following derivative of $Y = X^n$ using the Complicate Modulus M_n that fullfit the area below the derivative till an integer A, with Sum of Gnomons. in fact:

$$A^n = \sum_{x=1}^{A} [x^n - (x-1)^n]$$

So in the same way we did for $n = 2$ we can describe any Power of an Integers using a Sum of proper Ordinal Numbers I've called **n-th Gnomons** (as I told in the pre-face this are known as the Nexus's Numbers, but my definition is more detailed as you will discover). Where I define:

Complicate Modulus n-th the operator $M_n = [x^n - (x-1)^n]$

Gnomon each value: $M_{n,x_i} = [x^n - (x-1)^n]_{x=x_i} = [x_i^n - (x_i - 1)^n]$

I remember is important to left x as variable, or index, and i as the $i - th$ value of such variable.

The proof it's simple since this comes from the Binomial develop and from the most known Telescoping Sum property I already present here for $n = 2$. If we develop the sum we can immediately see that:

$$A^n = \sum_{x=1}^{A} [x^n - (x-1)^n] = A^n - (A-1)^n + (A-1)^n - (A-2)^n + (A-2)^n - ... + 1 - 1 = A^n$$

I call M_n =**Complicate Modulus** since it can be easily figured out from the classic modular arithmetic seeing that

it can cut any integer $P \in \mathbb{N}^+$ with rising slices of dimension

$$M_{n,x} = (x^n - (x-1)^n), \textbf{ Rest } = \mathbf{0}$$

if, and only if $P = A^n$ **;** $A \in \mathbb{N}^+$.

Table 3: Naturals rewritten via M_3 so as Cube plus Rest

x	M_3	Rest	
1	1	0	$= 1^3 + 0$
2	1	1	$= 1^3 + 1$
3	1	2	$= 1^3 + 2$
4	2	0	$= 1^3 + 3$
5	2	1	$= 1^3 + 4$
6	2	2	$= 1^3 + 5$
7	2	3	$= 1^3 + 6$
8	2	4	$= 2^3 + 0$
9	3	0	$= 2^3 + 1$
10	3	1	$= 2^3 + 2$
11	3	2	$= 2^3 + 3$
12	3	3	$= 2^3 + 4$
13	3	4	$= 2^3 + 5$

This holds true for A and n integers, but I'll show we can go over in $\mathbb{Q}^+$ in the next pages.

As example for $n = 3$, the term $M_n = [x^n - (x-1)^n]$ becomes:

$$M_3 = (3x^2 - 3x + 1)$$

To easily remember, keep Tartaglia's Terms for $(x-1)^3$, remove the first term and change the sign of the other, so we have:

$$A^3 = \sum_{x=1}^{A} (3x^2 - 3x + 1)$$

And so on for bigger n, following Tartaglia's triangle.

In the next pages I'll present again this case on a Cartesian Plane, hoping will be more clear (if necessary) why I use in all this work x as Index instead of the classic mute index m or i: in my representation it's, de-facto, the x coordinate on the Cartesian plane, and calling it 'mute' has encouraged peoples to discard to investigate more on it.

Complicate Modulus Algebra over X-Y Plane for $n > 2$:

What differs from $n = 2$ is that:

1) from $n = 3$ while continuous function $Y = X^n$ and its Continuous derivative $Y' = nX^n$ (and followings) are INVERTIBLE,

once we pass ($n > 2$) via the Integer derivative to the Integer Gnomons, we abandoning CONTINUITY and we Loose INVERTIBLE property.

2) again the Exceeding Area A^+ is equal to the Missing one A^- for the Telescoping Sum Property, and we can easy prove this.

3) Very important difference is that we LOSS, from $n > 2$ the same (triangular) Exceeding/Missing Area shape property, so they not just differ by size of the Bases and of the Heights,

- But they also have no longer the Same Shape at all, since the Left Missing Area A^- (Red one) has a Convex Lower Border, while the Right Exceeding A^+ (Blue one) has a Concave Upper Border.

And we will be also cleat that since the curvature of the First derivative, becomes Lower and Lower rising x (and or n) also the ratio r/q will change going closer and closer to 0.5 rising x and or n. Here one example on the picture where to distinguish the points I've Scaled Lot x/y to see the Curvature is in the real picture $x/y = 1$ very close to a line also for lower value of x and n.

Here the example of how a power $Y = X^3$, can be represented squaring its derivative $Y' = 3X^2$ via Gnomons (Red Columns width=1), following what I call Complicate Modulus Height (Black LINES):

We can for so represent a power of integer, i.e. 10^3

- as a point on the curve $y = x^3$, or as an area bellow its first derivative, or

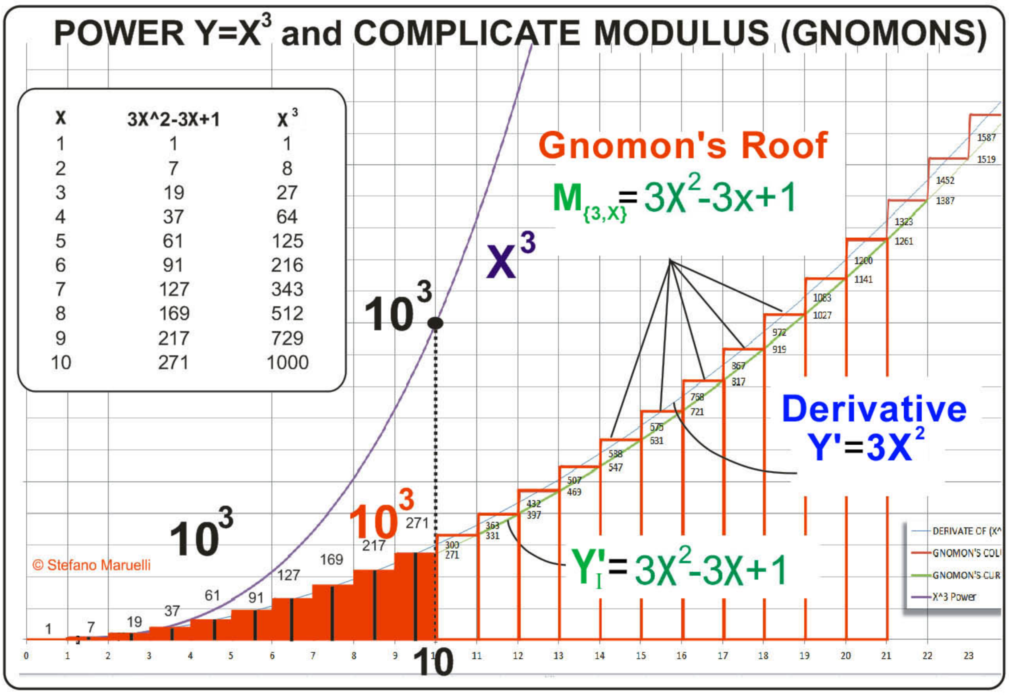
- as a Sum of Segments $M_3|_{x=i} = 3x^2 - 3x + 1$, for x from 1 to 10, that is also equal to the Sum of the Areas of the Gnomons $BASE = 1$ and height:

$$M_{3,i} = (3X^2 - 3X + 1)_{X=i} = 3i^2 - 3i + 1$$

Such Integer Gnomons always perfectly Square the derivative $Y' = nX^{n-1}$ for any $X \in N^*$.

This means that respect to the derivative $Y = 3X^2$ the Missing Area A^- on the Left is always Equal to the Exceeding Right one A^+.

As the Telescoping Sum, the Balancing Property also works for all the Following derivative.



An easy useful example of this property is given in the case $n = 3$ where the first derivative is $y' = 3x^2$ and where to find the balancing coordinate x_m we have to solve the equation:

$$3X_m^2 = 3X^2 - 3X + 1$$

As told all that works thanks to the Telescoping Sum Property.

The proof it's very easy developing the Sum (as made for $n = 2$) we have:

$$A^n = \sum_{X=1}^{A} (X^n - (X-1)^n) = A^n - (A-1)^n + (A-1)^n - (A-2)^n + (A-2)^n - ... + 1 - 1 = A^n$$

Here after I'll show as example also the case $n = 5$ where it's clear that:

In case $n > 2$ the derivative is a curve but the Balancing Property always holds.

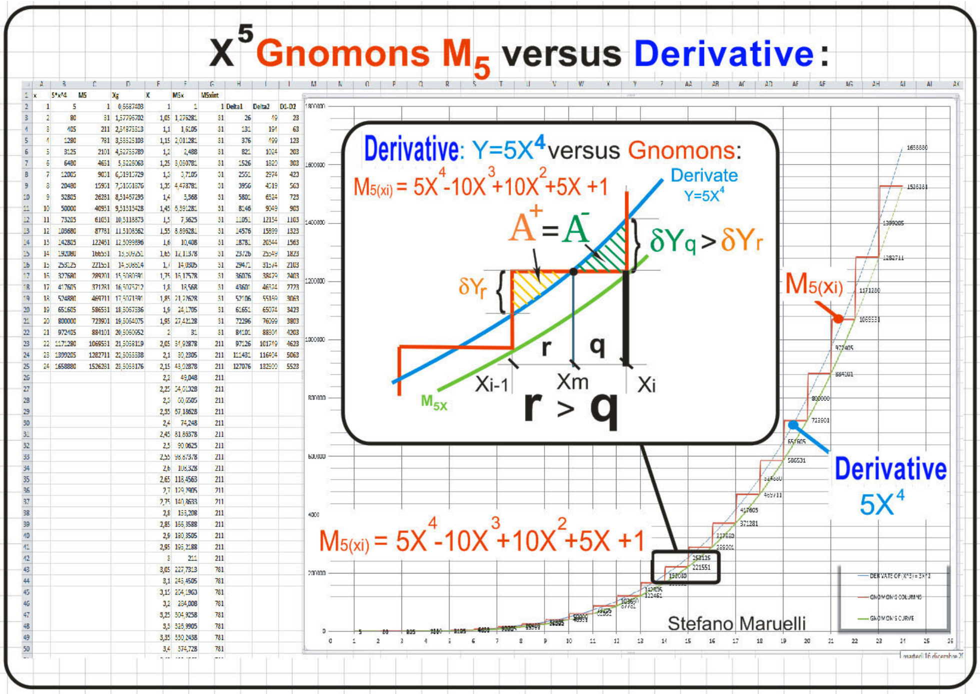
Summarizing we prove that the Sum of Gnomons Base=1, height:

$$M_n = (X^n - (X-1)^n)$$

calculated from 1 to an integer A,

is equal, in area, to the area below the (here in blue) derivative $y' = nX^{(n-1)}$ from 0 to A

I call the (here in green) continuous function $Y_I' = (X^n - (X-1)^n)$: the **Integer derivative** where it's clear it has nothing to do with the concept of the derivative since is a function that define the Right Upper corner, so the Height of the each Gnomon.

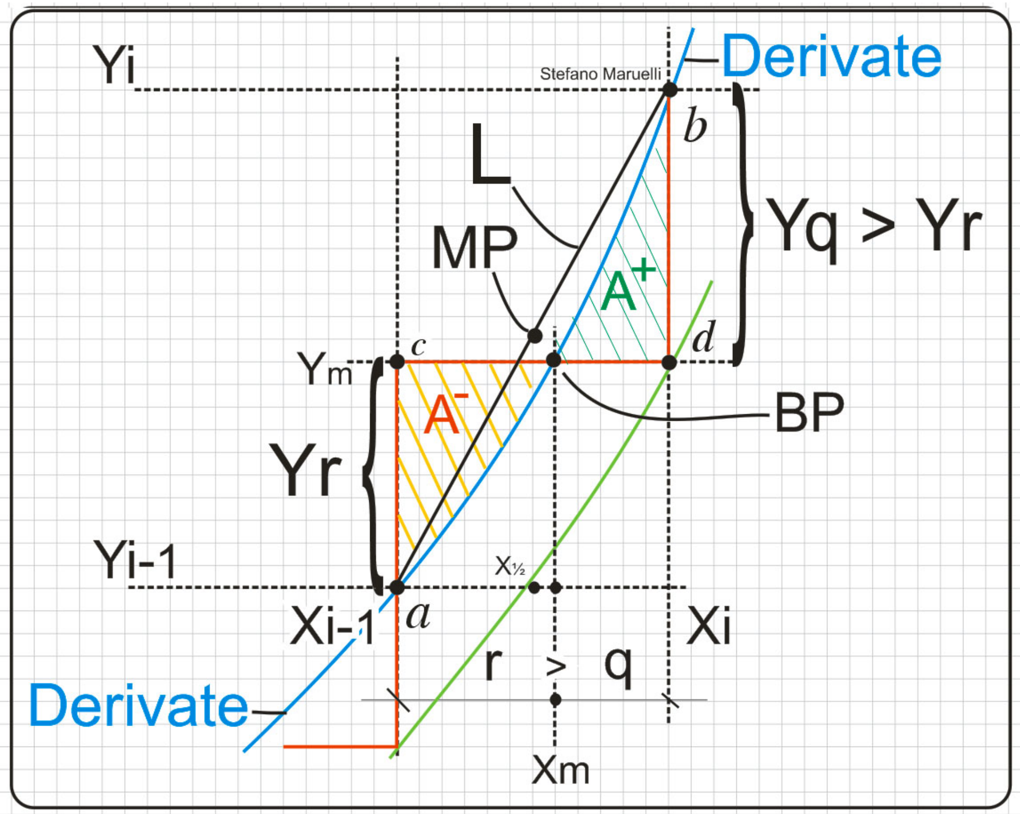


Telescoping Sum and The BALANCING POINT BP

All what I wrote till now is based on the Telescoping Sum Property, that can be seen as the capability of the curve of the type $y' = nx^{n-1}$ to be squared with Gnomons, so having a certain point, I've called the Balancing Point BP, that lies on the first derivative, and is the one for what: The Missing Left (Red) Area (A^-) will equate the Exceeding one (A^+).

We first start to calculate the Abscissa X_m that satisfy this condition, then we must now discover who is the **BALANCING POINT** BP , to prove it has Always IRRATIONAL coordinate $X_{m,i}$ if $n > 2$.

We aready know now that the Height $Y_{m,i}$ is the Integer (for now) Height of our Gnomons, so it is: $Y_{m,i} = M_{n,x_i}$ that is, for example in the case $n = 3$ equal to: $Y_{3,i} = 3X_i^2 - 3X_i + 1$



How to calculate $X_{m,i}$:

To calculate $x_{m,i}$ we have to write and solve the equation:$A^{+} = A^{-}$ and we can distinguish in two case: $n = 2$ so when the First Integer derivative $y' = 2X - 1$ is linear, and $n > 2$, so when the First Integer derivative $y' = (X^n - (X-1)^n)$ is a Curve.

For $n > 2$ the derivative is a curve, so we need to use the Integral to solve the Balancing Rule.

$$A^{+} = A^{-}$$

That becomes:

$$(X_{m,i} - X_{i-1}) * y_{m_i} - \int_{x_{i-1}}^{x_{m,i}} nx^{n-1} = \int_{x_{m,i}}^{x_i} nx^{n-1} - (x_i - x_{m,i}) * y_{m_i}$$

Still if we know how to solve it, we need to make many other concerning on our Complicate Modulus Algebra, before prove that $X_{m,i}$ is always an Irrational if $n > 2$ and that will be the goal of this first Vol.1.

After that we will have all the knowledge necessary to try to prove more complicate problems involving Power's od Integers, like Fermat the Last, but also of Rational.

We note that for $n > 2$ due to the Curved derivative, must be $X_m \neq \frac{X_{i-1}+X_i}{2}$ so must be $r > q$.

But to prove $X_{m,i}$ is always an Irrational if $n > 2$ I'll follow 2 ways, both will look in how BP is geometrically fixed.

- the first one involve simple concerning on the relative position of BP respect to known things: the Medium (or Center) Point MP,

- the second one will show that we can pack X_m between Two Following Integers, and then Between Two Following Rational depending by a factor $1/K$, and then we continuous ab infinitum this process, so we can push to the Limit the divisor K,

and just at that point we will rise X_m, so it can be just an Irrational. But to do this we need to show how we can play with Rational Gnomons, and this will require a new Chapters hereafter.

This process is known as Dedekind Cut. It sound like an Axiom, but we will prove it.

For now what we can immediately see is:

- We discover that due to the Telescoping Sum Property, without making other concerning than the one involving Integer Numbers and Proportional Areas, for any Parabolas $Y = aX^n$ in any derivative (also the following) the Exceeding Area A^+ will equate the Missing one A^-, ,

- But once we ask how much the value of such areas is the only way, for $n > 2$ to calculate them is to go infinitesimal and make the integral.

Comparison between Exceeding / Missing Areas $A_i^+ = A_i^-$:

Is clear that while below a Linear derivative $Y' = 2X$ following Exceeding/Missing Areas has always the same Area, independently by the Abscissa we choose, bellow a Curved derivative $y' = nX^{n-1}$ with $n > 2$ there is an Ordinal Rule between Such Areas, that all will converge to a Triangular Area once $n \to \infty$.

Calling A_i^+ the i-esim Exceeding Area, and A_i^- the i-esim Missing Area, below a curved derivative will always hold true this list of relations:

1) $A_i^+ = A_i^-$

2) $A_{i+k}^+ < A_i^-$ for any $k >= 1$ because going Right a Curved derivative becomes more Steep and Flat, so the Exceeding Area of the Next Gnomon will be littlest than the Previous Missing one. For the same reason will be true that:

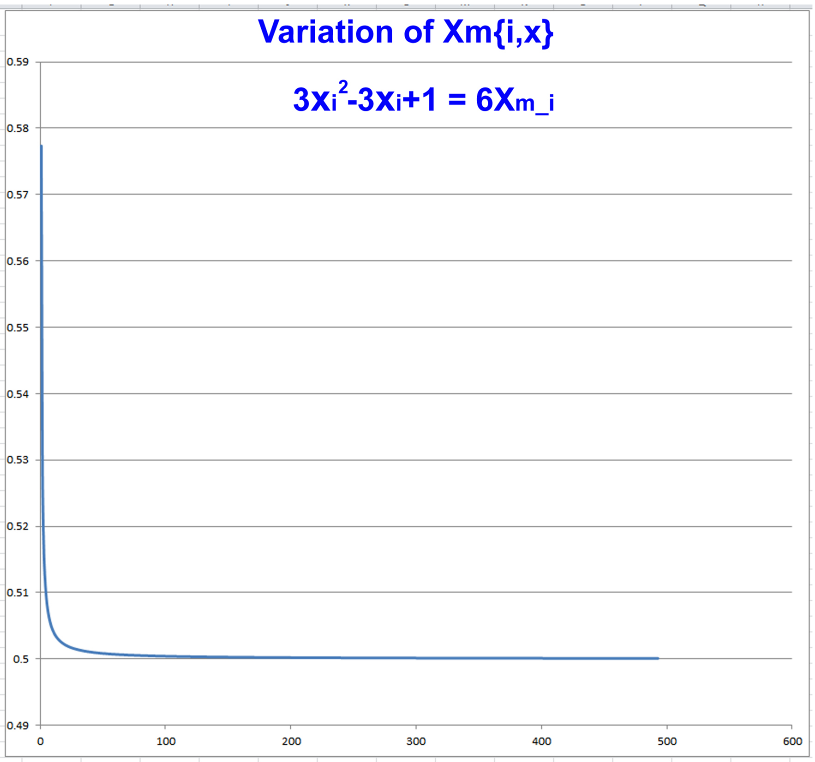
3) $A_{i+k}^- < A_i^-$ for any $k >= 1$

4) $A_{i+k}^- < A_i^-$ for any $k >= 1$

This property is in connection with the Fermat Last Theorem in a way will be explained in the Vol.2.

Here is how $X_{m,i}$ behave rising X, in the example $n = 3$:

Xi-1	Xi	Xm_i	r_i
1	2	0.577350269	0.577350269
2	3	1.527525232	0.527525232
3	4	2.516611478	0.516611478
4	5	3.511884584	0.511884584
5	6	4.509249753	0.509249753
6	7	5.507570547	0.507570547
7	8	6.506407099	0.506407099
8	9	7.505553499	0.505553499
9	10	8.504900548	0.504900548
10	11	9.504384953	0.504384953
11	12	10.5039675	0.503967504
12	12	11.50362262	0.503622618
13	12	12.50333289	0.503332889
14	12	13.50308607	0.503086067
15	12	14.50287328	0.502873279
16	12	15.50268794	0.502687939
17	12	16.50252506	0.502525059
18	12	17.50238079	0.50238079
19	12	18.50225212	0.502252115
20	12	19.50213664	0.502136635
21	12	20.50203242	0.50203242
22	12	21.5019379	0.501937897
23	12	22.50185178	0.501851776
24	12	23.50177298	0.501772983
25	12	24.50170062	0.501700621
26	12	25.50163393	0.501633935
27	12	26.50157228	0.50157228
28	12	27.50151511	0.50151511
29	12	28.50146195	0.501461951
30	12	29.5014124	0.501412396
31	12	30.50136609	0.50136609
32	12	31.50132272	0.501322724
33	12	32.50128203	0.501282026
34	12	33.50124376	0.501243758
35	12	34.50120771	0.501207708
36	12	35.50117369	0.50117369
37	12	36.50114153	0.501141535
38	12	37.50111109	0.501111095
39	12	38.50108224	0.501082236
40	12	39.50105484	0.501054838
41	12	40.50102879	0.501028794
42	12	41.501004	0.501004004
43	12	42.50098038	0.500980381
44	12	43.50095784	0.500957844
45	12	44.50093632	0.50093632
46	12	45.50091574	0.500915742



As you can see $X_{m,i}$ goes fast close to 0.5 and for higher n, due to the less curvature, we know it goes faster and faster close to 0.5.

The Scaling property: What will be very important to remember is that Scaling

on the Cartesian plane the representation of the curve $Y = X^n$ in X, Y or X, Y leaves unchanged the result.

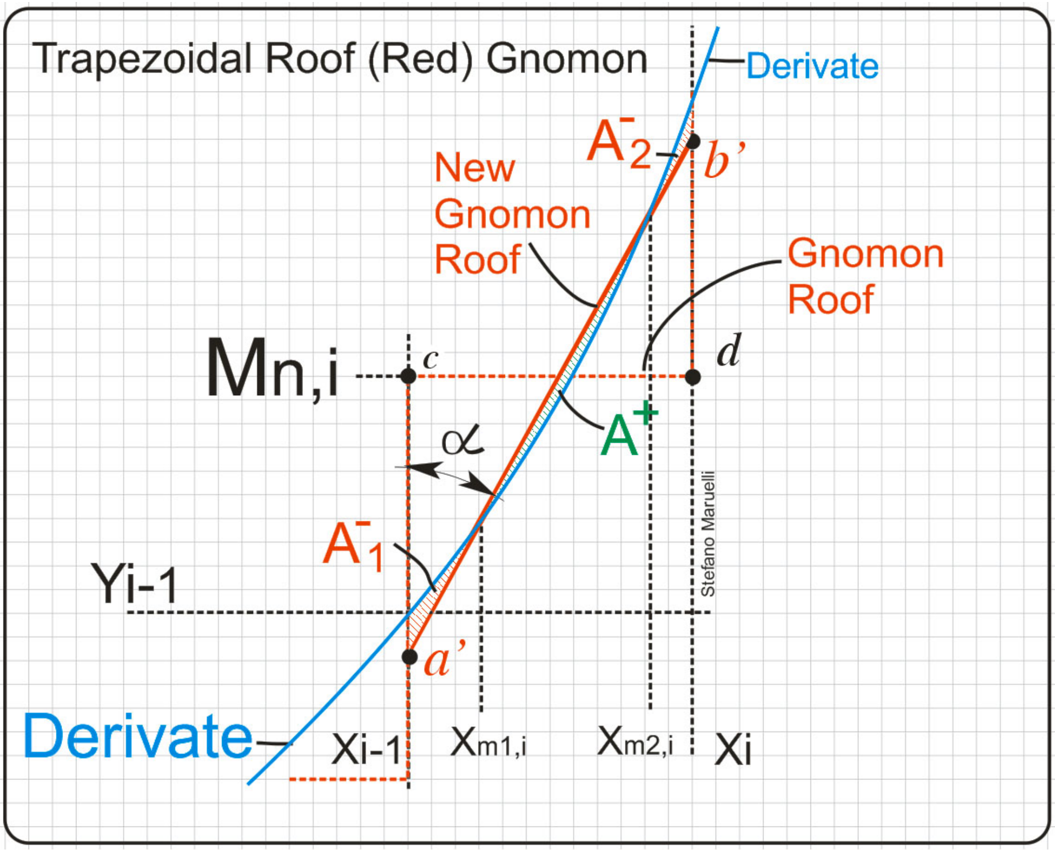
We do not give here a proof since is very simple and evident that if the Telescoping Sum Holds, then it Holds true independently by the scale factor X/Y we choose, so if we assume the same scale for X, and Y or we stretch one of them.

And this holds true also if we scale for example X of an irrational value f.e. $\sqrt{2}$. And this let us suspect we need to investigate more.

Trapezoidal Gnomons:

To open another Graphical parenthesis, for those understood this method, will be probably clear at this point that nobody denied us to keep, instead of a Rectangular Gnomons a Trapezoidal Gnomons, capable entirely cover the area bellow the derivative between two integers.

In the case $n > 2$ the Trapezium has to mediate the curve between the selected points as in the following picture. The equations bellow this new Gnomon is not included in this work since just question of make some more concerning on how to equate the Exceeding / Missing Areas and because the most simple way to find it is to remember we already define a flat roof Gnomon so is enough to 'turn' the roof around its medium point sure that in this way we holds the same area bellow the new non flat roof. Of course rotating around the center of any angle the area of the Gnomons rest the same, but there is a special angle for what the chain of roof becomes continuous, so it has interesting consequences for those who loves to find who are those 'special' $a_i' = b_{i-1}$ and $b_i' = a_{i+1}$ numbers let the chain.



While in the case the derivative is a Line the new Trapezoidal Gnomons can be exactly equal to the derivative, also in shape.

Avoiding to re-make the trivial concerning on how to equate the Exceeding / Missing Areas, it has interesting consequences for those loves special numbers.

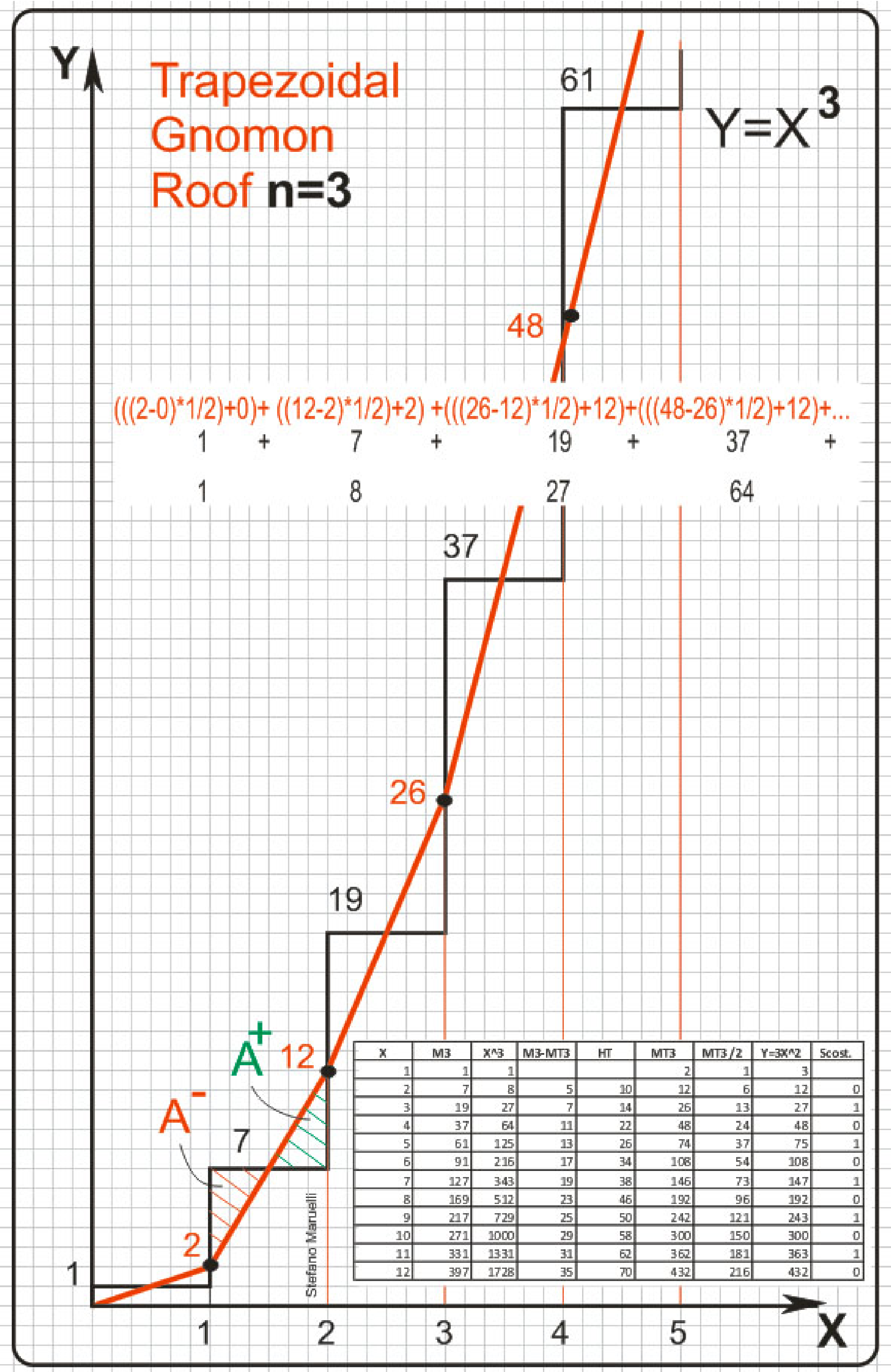
Here I'll present some case of interest, because the non trivial fact on what is possible to go on in the investigation is that the new Gnomon's Trapezoidal Roof is a Chain of Non Disjoint Segments, so another "magic" property was discovered:

the difference of height between Yi on the derivative and the new eight a'_{i+i} that fix the point where the new Linear Gnomon's Roof X_{i+1} starts, is exactly equal to the distance between Yi and b'_i so the point where the previous Gnomon's Linear Roof was stopped.

Here an example of Trapezoidal Gnomons Roof: the known values $1, 7, 19, 3x^2 - 3x + 1$ can be rebuild by trapezoid using the simple concept to have a triangular top that hold the equality $A^+ = A^-$, and a rectangular base. New series of numbers was added in Oeis.org by the author.

A broken line define an area that it's equal to the one bellow the derivative. Of course this is not the only possible one.

In the next page the table of the Trapezoidal Gnomons $n = 3, 4, 5, 6$



X	M3	X^3	M3-MT3	HT	MT3	MT3 /2	Y=3X^2	Scost.
1	1	1			2	1	3	
2	7	8	5	10	12	6	12	0
3	19	27	7	14	26	13	27	1
4	37	64	11	22	48	24	48	0
5	61	125	13	26	74	37	75	1
6	91	216	17	34	108	54	108	0
7	127	343	19	38	146	73	147	1
8	169	512	23	46	192	96	192	0
9	217	729	25	50	242	121	243	1
10	271	1000	29	58	300	150	300	0
11	331	1331	31	62	362	181	363	1
12	397	1728	35	70	432	216	432	0

Several know sequence are hidden behind this new recursive formula to obtain the Trapezoidal Gnomons. The first interesting, for $n = 3$ is the sequence known as $A032528$: Concentric hexagonal numbers: $\text{floor}(3 * n^2/2)$.

X	M3	M3-MT3	HT	MT3	MT3 /2	Y=3X^2	Scost.
1	1			2	1	3	
2	7	5	10	12	6	12	0
3	19	7	14	26	13	27	1
4	37	11	22	48	24	48	0
5	61	13	26	74	37	75	1
6	91	17	34	108	54	108	0
7	127	19	38	146	73	147	1
8	169	23	46	192	96	192	0
9	217	25	50	242	121	243	1
10	271	29	58	300	150	300	0
11	331	31	62	362	181	363	1
12	397	35	70	432	216	432	0
13	469	37	74	506	253	507	1
14	547	41	82	588	294	588	0
15	631	43	86	674	337	675	1
16	721	47	94	768	384	768	0
17	817	49	98	866	433	867	1
18	919	53	106	972	486	972	0
19	1027	55	110	1082	541	1083	1
20	1141	59	118	1200	600	1200	0
21	1261	61	122	1322	661	1323	1
22	1387	65	130	1452	726	1452	0
23	1519	67	134	1586	793	1587	1
24	1657	71	142	1728	864	1728	0
25	1801	73	146	1874	937	1875	1
26	1951	77	154	2028	1014	2028	0
27	2107	79	158	2186	1093	2187	1
28	2269	83	166	2352	1176	2352	0
29	2437	85	170	2522	1261	2523	1
30	2611	89	178	2700	1350	2700	0
31	2791	91	182	2882	1441	2883	1
32	2977	95	190	3072	1536	3072	0
33	3169	97	194	3266	1633	3267	1
34	3367	101	202	3468	1734	3468	0
35	3571	103	206	3674	1837	3675	1
36	3781	107	214	3888	1944	3888	0
37	3997	109	218	4106	2053	4107	1
38	4219	113	226	4332	2166	4332	0
39	4447	115	230	4562	2281	4563	1
40	4681	119	238	4800	2400	4800	0
41	4921	121	242	5042	2521	5043	1
42	5167	125	250	5292	2646	5292	0

X	M4	M4-MT4	HT	MT4	MT4/2	Y=4X^3	Scost.
1	1			2	1	4	4
2	15	13	26	28	14	32	6
3	65	37	74	102	51	108	34
4	175	73	146	248	124	256	110
5	369	121	242	490	245	500	258
6	671	181	362	852	426	864	502
7	1105	253	506	1358	679	1372	866
8	1695	337	674	2032	1016	2048	1374
9	2465	433	866	2898	1449	2916	2050
10	3439	541	1082	3980	1990	4000	2918
11	4641	661	1322	5302	2651	5324	4002
12	6095	793	1586	6888	3444	6912	5326
13	7825	937	1874	8762	4381	8788	6914
14	9855	1093	2186	10948	5474	10976	8790
15	12209	1261	2522	13470	6735	13500	10978
16	14911	1441	2882	16352	8176	16384	13502
17	17985	1633	3266	19618	9809	19652	16386
18	21455	1837	3674	23292	11646	23328	19654
19	25345	2053	4106	27398	13699	27436	23330
20	29679	2281	4562	31960	15980	32000	27438
21	34481	2521	5042	37002	18501	37044	32002
22	39775	2773	5546	42548	21274	42592	37046
23	45585	3037	6074	48622	24311	48668	42594
24	51935	3313	6626	55248	27624	55296	48670
25	58849	3601	7202	62450	31225	62500	55298
26	66351	3901	7802	70252	35126	70304	62502
27	74465	4213	8426	78678	39339	78732	70306
28	83215	4537	9074	87752	43876	87808	78734
29	92625	4873	9746	97498	48749	97556	87810
30	102719	5221	10442	107940	53970	108000	97558
31	113521	5581	11162	119102	59551	119164	108002
32	125055	5953	11906	131008	65504	131072	119166
33	137345	6337	12674	143682	71841	143748	131074
34	150415	6733	13466	157148	78574	157216	143750
35	164289	7141	14282	171430	85715	171500	157218
36	178991	7561	15122	186552	93276	186624	171502
37	194545	7993	15986	202538	101269	202612	186626
38	210975	8437	16874	219412	109706	219488	202614
39	228305	8893	17786	237198	118599	237276	219490
40	246559	9361	18722	255920	127960	256000	237278
41	265761	9841	19682	275602	137801	275684	256002
42	285935	10333	20666	296268	148134	296352	275686

X	M_5	M5-MT5	HT	MT5	MT5/2	Y=5X^4	Scost.
1	1	1		2	1	5	4
2	31	29	58	60	30	80	49
3	211	151	302	362	181	405	194
4	781	419	838	1200	600	1280	499
5	2101	901	1802	3002	1501	3125	1024
6	4651	1649	3298	6300	3150	6480	1829
7	9031	2731	5462	11762	5881	12005	2974
8	15961	4199	8398	20160	10080	20480	4519
9	26281	6121	12242	32402	16201	32805	6524
10	40951	8549	17098	49500	24750	50000	9049
11	61051	11551	23102	72602	36301	73205	12154
12	87781	15179	30358	102960	51480	103680	15899
13	122461	19501	39002	141962	70981	142805	20344
14	166531	24569	49138	191100	95550	192080	25549
15	221551	30451	60902	252002	126001	253125	31574
16	289201	37199	74398	326400	163200	327680	38479
17	371281	44881	89762	416162	208081	417605	46324
18	469711	53549	107098	523260	261630	524880	55169
19	586531	63271	126542	649802	324901	651605	65074
20	723901	74099	148198	798000	399000	800000	76099
21	884101	86101	172202	970202	485101	972405	88304
22	1069531	99329	198658	1168860	584430	1171280	101749
23	1282711	113851	227702	1396562	698281	1399205	116494
24	1526281	129719	259438	1656000	828000	1658880	132599
25	1803001	147001	294002	1950002	975001	1953125	150124
26	2115751	165749	331498	2281500	1140750	2284880	169129
27	2467531	186031	372062	2653562	1326781	2657205	189674
28	2861461	207899	415798	3069360	1534680	3073280	211819
29	3300781	231421	462842	3532202	1766101	3536405	235624
30	3788851	256649	513298	4045500	2022750	4050000	261149
31	4329151	283651	567302	4612802	2306401	4617605	288454
32	4925281	312479	624958	5237760	2618880	5242880	317599
33	5580961	343201	686402	5924162	2962081	5929605	348644
34	6300031	375869	751738	6675900	3337950	6681680	381649
35	7086451	410551	821102	7497002	3748501	7503125	416674
36	7944301	447299	894598	8391600	4195800	8398080	453779
37	8877781	486181	972362	9363962	4681981	9370805	493024
38	9891211	527249	1054498	10418460	5209230	10425680	534469
39	10989031	570571	1141142	11559602	5779801	11567205	578174
40	12175801	616199	1232398	12792000	6396000	12800000	624199
41	13456201	664201	1328402	14120402	7060201	14128805	672604
42	14835031	714629	1429258	15549660	7774830	15558480	723449

X	M_6	M6-MT6	HT	MT_6	MT_6/2	Y=6X^5	Scost.
1	1			2	1	6	5
2	63	61	122	124	62	192	190
3	665	541	1082	1206	603	1458	1455
4	3367	2161	4322	5528	2764	6144	6140
5	11529	6001	12002	17530	8765	18750	18745
6	31031	13501	27002	44532	22266	46656	46650
7	70993	26461	52922	97454	48727	100842	100835
8	144495	47041	94082	191536	95768	196608	196600
9	269297	77761	155522	347058	173529	354294	354285
10	468559	121501	243002	590060	295030	600000	599990
11	771561	181501	363002	953062	476531	966306	966295
12	1214423	261361	522722	1475784	737892	1492992	1492980
13	1840825	365041	730082	2205866	1102933	2227758	2227745
14	2702727	496861	993722	3199588	1599794	3226944	3226930
15	3861089	661501	1323002	4522590	2261295	4556250	4556235
16	5386591	864001	1728002	6250592	3125296	6291456	6291440
17	7360353	1109761	2219522	8470114	4235057	8519142	8519125
18	9874655	1404541	2809082	11279196	5639598	11337408	11337390
19	13033657	1754461	3508922	14788118	7394059	14856594	14856575
20	16954119	2166001	4332002	19120120	9560060	19200000	19199980
21	21766121	2646001	5292002	24412122	12206061	24504606	24504585
22	27613783	3201661	6403322	30815444	15407722	30921792	30921770
23	34655985	3840541	7681082	38496526	19248263	38618058	38618035
24	43067087	4570561	9141122	47637648	23818824	47775744	47775720
25	53037649	5400001	10800002	58437650	29218825	58593750	58593725
26	64775151	6337501	12675002	71112652	35556326	71288256	71288230
27	78504713	7392061	14784122	85896774	42948387	86093442	86093415
28	94469815	8573041	17146082	103042856	51521428	1.03E+08	1.03E+08
29	112933017	9890161	19780322	122823178	61411589	1.23E+08	1.23E+08
30	134176679	11353501	22707002	145530180	72765090	1.46E+08	1.46E+08
31	158503681	12973501	25947002	171477182	85738591	1.72E+08	1.72E+08
32	186238143	14760961	29521922	200999104	100499552	2.01E+08	2.01E+08
33	217726145	16727041	33454082	234453186	117226593	2.35E+08	2.35E+08
34	253336447	18883261	37766522	272219708	136109854	2.73E+08	2.73E+08
35	293461209	21241501	42483002	314702710	157351355	3.15E+08	3.15E+08
36	338516711	23814001	47628002	362330712	181165356	3.63E+08	3.63E+08
37	388944073	26613361	53226722	415557434	207778717	4.16E+08	4.16E+08
38	445209975	29652541	59305082	474862516	237431258	4.75E+08	4.75E+08
39	507807377	32944861	65889722	540752238	270376119	5.41E+08	5.41E+08
40	577256239	36504001	73008002	613760240	306880120	6.14E+08	6.14E+08
41	654104241	40344001	80688002	694448242	347224121	6.95E+08	6.95E+08
42	738927503	44479261	88958522	783406764	391703382	7.84E+08	7.84E+08

A032528 Concentric hexagonal numbers: floor(3*n^2 / 2).

+400
43

0, 1, 6, 13, 24, 37, 54, 73, 96, 121, 150, 181, 216, 253, 294, 337, 384, 433, 486, 541, 600, 661, 726, 793, 864, 937, 1014, 1093, 1176, 1261, 1350, 1441, 1536, 1633, 1734, 1837, 1944, 2053, 2166, 2281, 2400, 2521, 2646, 2773, 2904, 3037, 3174, 3313, 3456, 3601, 3750 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,3

COMMENTS
From Omar E. Pol, Aug 20 2011: (Start)
Also, cellular automaton on the hexagonal net. The sequence gives the number of "ON" cells in the structure after n-th stage. A007310 gives the first differences. For a definition without words see the illustration of initial terms in the example section. Note that the cells become intermittent. A083577 gives the primes of this sequences.
Also, A033581 and A003154 interleaved.
Also, row sums of an infinite square array T(n,k) in which column k lists 2*k-1 zeros followed by the numbers A008458 (see example). (End)
Sequence found by reading the line from 0, in the direction 0, 1,... and the same line from 0, in the direction 0, 6,..., in the square spiral whose vertices are the generalized pentagonal numbers A001318. Main axis perpendicular to A045943 in the same spiral. - Omar E. Pol, Sep 08 2011

LINKS
Vincenzo Librandi, Table of n, a(n) for n = 0..10000
Index entries for sequences related to cellular automata
Index entries for linear recurrences with constant coefficients, signature (2,0,-2,1).

FORMULA
G.f.: (x+4*x^2+x^3)/(1-2*x+2*x^3-x^4) = x*(1+4*x+x^2)/((1+x)*(1-x)^3).
a(n) = +2*a(n-1) -2*a(n-3) +1*a(n-4). - Joerg Arndt, Aug 22 2011
a(n) = (6*n^2+(-1)^n-1)/4. - Bruno Berselli, Aug 22 2011
a(n) = A184533(n), n >= 2. - Clark Kimberling, Apr 20 2012
First differences of A011934: a(n) = A011934(n) - A011934(n-1) for n>0. - Franz Vrabec, Feb 17 2013

EXAMPLE
From Omar E. Pol, Aug 20 2011: (Start)
Using the numbers A008458 we can write:
```
0, 1, 6, 12, 18, 24, 30, 36, 42,  48,  54, ...
0, 0, 0,  1,  6, 12, 18, 24, 30,  36,  42, ...
0, 0, 0,  0,  0,  1,  6, 12, 18,  24,  30, ...
0, 0, 0,  0,  0,  0,  0,  1,  6,  12,  18, ...
0, 0, 0,  0,  0,  0,  0,  0,  0,   1,   6, ...
```
And so on.
===
The sums of the columns give this sequence:
0, 1, 6, 13, 24, 37, 54, 73, 96, 121, 150, ...
...
Illustration of initial terms as concentric hexagons:
```
.
.                                         o o o o o
.                         o o o o        o         o
.             o o o      o       o      o   o o o   o
.     o o    o     o    o   o o   o    o   o     o   o
. o  o   o  o   o   o  o   o   o   o  o   o   o   o   o
.     o o    o     o    o   o o   o    o   o     o   o
.             o o o      o       o      o   o o o   o
.                         o o o o        o         o
.                                         o o o o o
.
. 1    6        13           24                37
.
```
(End)

MATHEMATICA
f[n_, m_] := Sum[Floor[n^2/k], {k, 1, m}]; t = Table[f[n, 2], {n, 1, 90}] (* Clark Kimberling, Apr 20 2012 *)

PROG
(MAGMA) [Floor(3*n^2/2): n in [0..50]]; // Vincenzo Librandi, Aug 21 2011
(Haskell)
```
a032528 n = a032528_list !! n
a032528_list = scanl (+) 0 a007310_list
-- Reinhard Zumkeller, Jan 07 2012
```
(PARI) a(n)=3*n^2\2 \\ Charles R Greathouse IV, Sep 24 2015

CROSSREFS
Cf. A003154, A007310, A008458, A033581, A083577, A000326, A001318, A005449, A045943, A032527, A195041. Column 6 of A195040.

AUTHOR
N. J. A. Sloane.

For n=4, is the sequence known as $A007588$: Stella Octangula numbers: $a(n) = n * (2 * n^2 - 1)$.

A007588 Stella octangula numbers: a(n) = n*(2*n^2 - 1). +440 32
(Formerly M4932)

0, 1, 14, 51, 124, 245, 426, 679, 1016, 1449, 1990, 2651, 3444, 4381, 5474, 6735, 8176, 9809, 11646, 13699, 15980, 18501, 21274, 24311, 27624, 31225, 35126, 39339, 43876, 48749, 53970, 59551, 65504, 71841, 78574, 85715, 93276, 101269, 109706, 118599, 127960

(list; graph; refs; listen; history; text; internal format)

OFFSET 0,3

COMMENTS
Also as a(n)=(1/6)*(12*n^3-6*n), n>0: structured hexagonal anti-diamond numbers (vertex structure 13) (Cf. A005915 = alternate vertex; A100188 = structured anti-diamonds; A100145 for more on structured numbers). - James A. Record (james.record(AT)gmail.com), Nov 07 2004
The only known square stella octangula number for n>1 is a(169) = 169*(2*169^2 - 1) = 9653449 = 3107^2. - Alexander Adamchuk, Jun 02 2008
Ljunggren proved that 9653449 = (13*239)^2 is the only square stella octangula number for n>1. See A229384 and the Wikipedia link. - Jonathan Sondow, Sep 30 2013
4*A007588 = A144138(ChebyshevU[3,n]). - Vladimir Joseph Stephan Orlovsky, Jun 30 2011
If A016813 is regarded as a regular triangle (with leading terms listed in A001844), a(n) provides the row sums of this triangle: 1, 5+9=14, 13+17+21=51 and so on. - J. M. Bergot, Jul 05 2013
Shares its digital root, A267017, with n*(n^2 + 1)/2 ("sum of the next n natural numbers" see A006003). - Peter M. Chema, Aug 28 2016

REFERENCES
J. H. Conway and R. K. Guy, The Book of Numbers, Copernicus Press, NY, 1996, p. 51.
E. Deza and M. M. Deza, Figurate numbers, World Scientific Publishing (2012), page 140.
W. Ljunggren, Zur Theorie der Gleichung x^2 + 1 = Dy^4, Avh. Norske Vid. Akad. Oslo. I. 1942 (5): 27.
N. J. A. Sloane and Simon Plouffe, The Encyclopedia of Integer Sequences, Academic Press, 1995 (includes this sequence).

LINKS
Alexander Adamchuk and Vincenzo Librandi, Table of n, a(n) for n = 0..10000 [Alexander Adamchuk computed terms 0 - 169, Jun 02, 2008; Vincenzo Librandi computed the first 10000 terms, Aug 18 2011]
A. Bremner, R. Høibakk, D. Lukkassen, Crossed ladders and Euler's quartic, Annales Mathematicae et Informaticae, 36 (2009) pp. 29-41. See p. 33.
T. P. Martin, Shells of atoms, Phys. Reports, 273 (1996), 199-241, eq. (11).
Eric Weisstein's World of Mathematics, Stella Octangula Number
Wikipedia, Stella octangula number
Index entries for linear recurrences with constant coefficients, signature (4,-6,4,-1).

FORMULA
G.f.: x*(1+10*x+x^2)/(1-x)^4.
a(n) = n*A056220(n).
a(n) = 4*a(n-1) - 6*a(n-2) + 4*a(n-3) - a(n-4), n>3. - Harvey P. Dale, Sep 16 2011
From Ilya Gutkovskiy, Jul 02 2016: (Start)
E.g.f.: x*(1 + 6*x + 2*x^2)*exp(x).
Dirichlet g.f.: 2*zeta(s-3) - zeta(s-1). (End)
a(n) = A004188(n) + A135503(n). - Miguel Cerda, Dec 25 2016

MAPLE
A007588:=n->n*(2*n^2 - 1); seq(A007588(n), n=0..40); # Wesley Ivan Hurt, Mar 10 2014

MATHEMATICA
Table[n(2n^2-1), {n, 0, 169}] (* Alexander Adamchuk, Jun 02 2008 *)
LinearRecurrence[{4, -6, 4, -1}, {0, 1, 14, 51}, 50] (* Harvey P. Dale,

PROG
(PARI) a(n)=n*(2*n^2-1)
(MAGMA) [n*(2*n^2 - 1): n in [0..40]]; // Vincenzo Librandi, Aug 18 2011

CROSSREFS
Backwards differences give star numbers A003154: A003154(n)=A007588(n)-A007588(n-1).
1/12*t*(n^3-n)+ n for t = 2, 4, 6, ... gives A004006, A006527, A006003, A005900, A004068, A000578, A004126, A000447, A004188, A004466, A004467, A007588, A062025, A063521, A063522, A063523.
Cf. A001653 = Numbers n such that 2*n^2 - 1 is a square.
a(169) = (A229384(3)*A229384(4))^2.
Cf. A267017, A006003.

AUTHOR
N. J. A. Sloane

EXTENSIONS
In the formula given in the 1995 Encyclopedia of Integer Sequences, the second 2 should be an exponent.

Chapt.4. A simple algorithm for finding the n-th Root

To extract the n-th root of a number P we can make the inverse process, so a Recursive Difference we can indicate with the Greek letter δ.

Once we chose the n-th Complicate Modulus we are interested in, for example $n = 3$, so M_3, we can have 2 main cases:

1) The recursive difference will give us back an (also called: n-th Power Zero)

2) The recursive difference will give us back an Integer plus a

if $P = A^n$ here for example $P = A^3 = 27$ than starting from P if we remove the following Gnomons $M_3 = 3x^2 - 3x + 1$ starting from $x = 1$ we can make the n-th root (by hand) in this simple way:

$$27 - (3x^2 - 3x + 1)_{x=1} = 27 - 1 = 26$$

$$26 - (3x^2 - 3x + 1)_{x=2} = 26 - 7 = 19$$

$$19 - (3x^2 - 3x + 1)_{x=3} = 19 - 19 = 0$$

So we can write 27 **as:** $27 = 3M_3 + 0$

This is a very slow algorithm to extract any n-th root also by hand, so it has no interest for computation; but it's clear it can sieve each Integer (for now) Number P to have back its Integer (or not) n-th root.

if $P = A^n$ here for example $P = 28$ then starting from P if we remove the followings Gnomons $M_3 = 3x^2 - 3x + 1$ starting from $x = 1$ we can make the n-th root p (by hand) in this simple way:

$$28 - (3x^2 - 3x + 1)_{x=1} = 28 - 1 = 27$$

$$27 - (3x^2 - 3x + 1)_{x=2} = 27 - 7 = 20$$

$$20 - (3x^2 - 3x + 1)_{x=3} = 20 - 19 = 1$$

So we can write 28 **as:** $28 = 3M_3 + 1$

Each time we have a Rest we are sure that our integer P is not a Power of an integer.

As we will see in the next chapters we can say more in case we have a Rest

We can therefore use the right Complicate Modulus M_n for the $n-th$ *Root* we desire and δ symbol to indicate this Recursive Difference Process we will do, where $P \in \mathbb{N}$ is any number and the Integer Upper Limit p is also our Integer Unknown Root (so the variable since it's the result we are looking for) :

So if:

$$P + \delta_{x=1}^{\lfloor P \rfloor}(x^n - (x-1)^n) = 0$$

Than $P = p^n; p \in \mathbb{N}$, so $Rest = 0$. Vice versa if:

$$P + \delta_{x=1}^{\lfloor P \rfloor}(x^n - (x-1)^n) > 0$$

Than $P \neq p^n; p \in \mathbb{N}$, so we have a $Rest = P - \delta_{x=1}^{\lfloor P \rfloor}(x^n - (x-1)^n)$

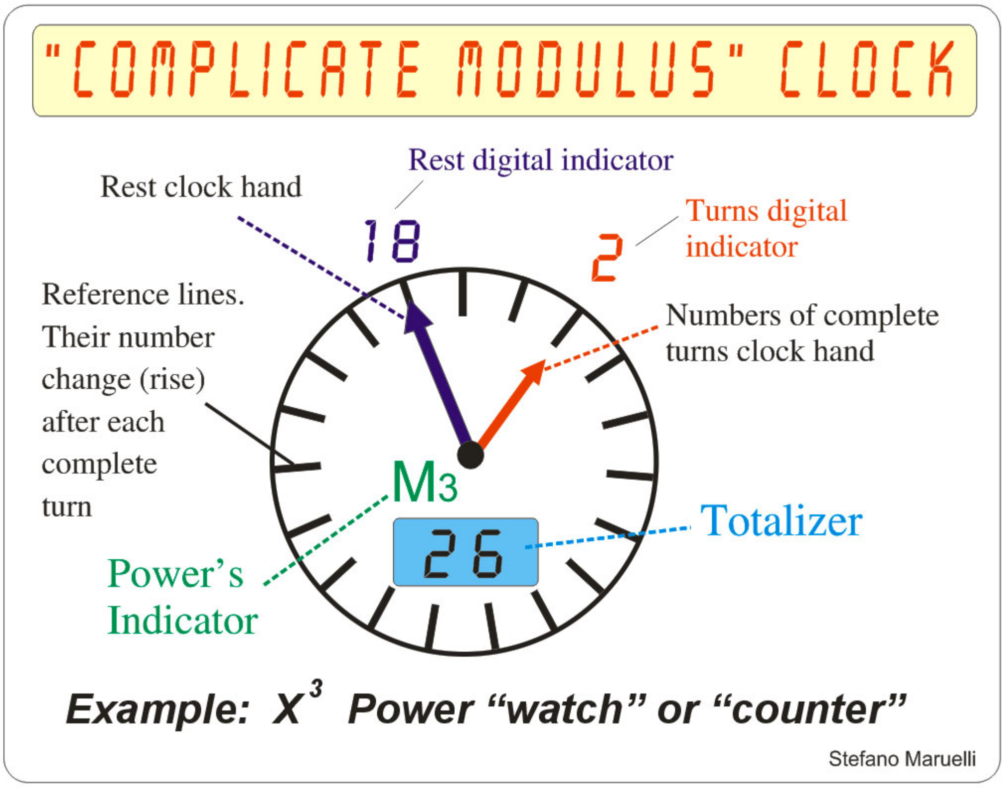
As for Modular Algebra is possible to compare it to a One Hand clock, we have now all the info to invent a new Two-Hand-Clock, much powerful than the old one. Going deeper in this concept several new simple but interesting properties will be found (and I think I've opened a new branch of math to young students not jet ready for Group theory / Abstract Algebra.

I can imagine many of you will jump on the chair once, in a few pages, when I'll hack the Sum Operator, but, again, you will see will be for a good reason.

Chapt.5 The Two Hands Clock

Modular Algebra is based on the well known One-Hand-Clock and just let us know the right hour for 1 second, two times each day.

Here I present my new Two-Hand-Clock, having a Digital Display that always show the right hour, base the n-th Power n we decide:



The Two-Hand-Clock show us more information than the classic single hand. It display unambiguously any Integer P as a Complicate Number, as the previous definition I gave. For the moment I present how it shows the Reduced Integer Complicate Numbers:

$$P = \lfloor P^{1/n} \rfloor^n + \left(P - \lfloor P^{1/n} \rfloor\right)^n$$

The Two-Hand-Clock, for Integers has some special characteristics:

- A digital display that show the number P (I've called: Tote)

- 2 hands, for the moment we suppose moving not continuously, but jumping from a Reference Line to another:

- One short (Red) for the Hours, that show p the Integer Root of P

- One Long (blue) for the minutes, that here show the $REST = R$

- A digital background able to shown the Reference Lines, that change each complete turns of the short hand, and will be useful to fix the scale for 'analog showing' the two information: the Integer Root p and the Rest R.

The number of the divisions (Reference Lines) rise each turn. It's the value of the Complicate Modulus M_n calculated for the current Actual Integer Root $p+1$, value.

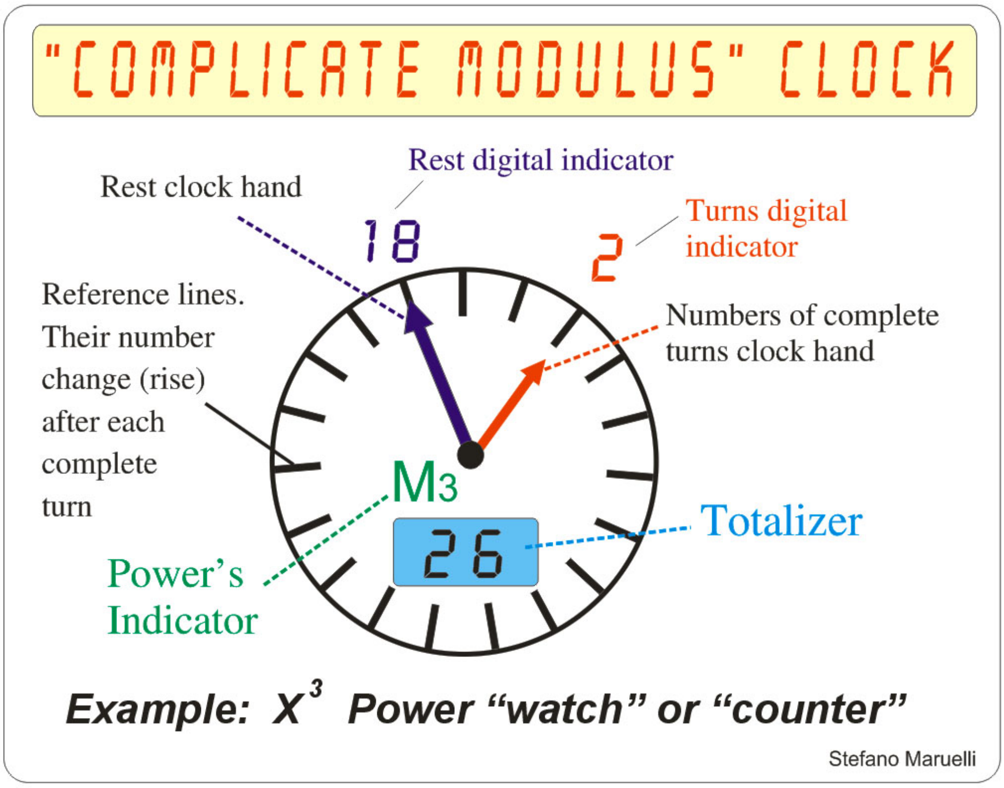
- A 2 digit display showing what n-th Power n we are using root; here in the previous picture, the Cubic one, so $n = 3$.

- To let more easy to read the position of the two hands, there are also over the clock two Digital Numbers moving with the Hands:

- one Red showing the Integer Root p, and

- one Blue showing the value of the Rest R

Here again the picture:

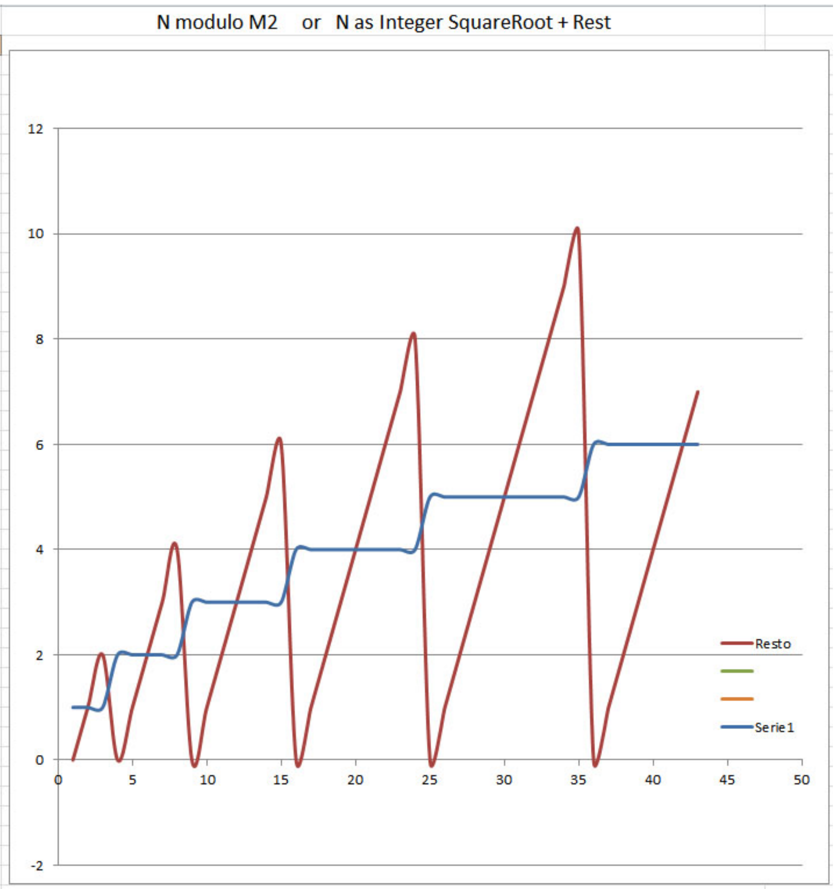


You can find animated Gif, upgrade and other info at my web page:

http : //shoppc.maruelli.com/prime$_s$tudy.htm

We can also tabulate the Integer P, the Integer Root and the Rest, to graphically show what happen to this value.

P	M2	Resto
1	1	0
2	1	1
3	1	2
4	2	0
5	2	1
6	2	2
7	2	3
8	2	4
9	3	0
10	3	1
11	3	2
12	3	3
13	3	4
14	3	5
15	3	6
16	4	0
17	4	1
18	4	2
19	4	3
20	4	4
21	4	5
22	4	6
23	4	7
24	4	8
25	5	0
26	5	1
27	5	2
28	5	3
29	5	4
30	5	5
31	5	6
32	5	7
33	5	8
34	5	9
35	5	10
36	6	0
37	6	1
38	6	2
39	6	3
40	6	4
41	6	5
42	6	6



Here an example of how the 2 hands clock show $P \in N$ in terms of an $IntegerSquareRoot$ and a $Rest$

Using the right Complicate Modulus $M_n = (x^n - (x-1)^n)$ it's possible to show any $P \in N^+$ Base any n-th Power we decide.

Will be interesting to think to a more evolved version where the two arrows moves continuously as the standard clock, since in this way we can also show any $P \in \mathbb{R}$.

Investigating in the Properties of the Rest

Is time now to better investigate in the Properties of the Rest since is a well known branch of math.

The Sign in front of the Rest:

First of all I assume till here that the Rest is Positive, while is clear that it can be also Negative, in fact we can also make another bijection:

$P = pM_2 + Rest_+ = (p+1)M_2 + Rest_- = (p+1)M_2 - |Rest_-|$

P	IR (1/2)	Rest	IR+1	Rest	P	IR (1/3)	Rest	IR+1	Rest
1	1	0	2	-3	1	1	0	2	-7
2	1	1	2	-2	2	1	1	2	-6
3	1	2	2	-1	3	1	2	2	-5
4	2	0	3	-5	4	1	3	2	-4
5	2	1	3	-4	5	1	4	2	-3
6	2	2	3	-3	6	1	5	2	-2
7	2	3	3	-2	7	1	6	2	-1
8	2	4	3	-1	8	2	0	3	-19
9	3	0	4	-7	9	2	1	3	-18
10	3	1	4	-6	10	2	2	3	-17
11	3	2	4	-5	11	2	3	3	-16
12	3	3	4	-4	12	2	4	3	-15
13	3	4	4	-3	13	2	5	3	-14
14	3	5	4	-2	14	2	6	3	-13
15	3	6	4	-1	15	2	7	3	-12
16	4	0	5	-9	16	2	8	3	-11
17	4	1	5	-8	17	2	9	3	-10
18	4	2	5	-7	18	2	10	3	-9
19	4	3	5	-6	19	2	11	3	-8
20	4	4	5	-5	20	2	12	3	-7
21	4	5	5	-4	21	2	13	3	-6
22	4	6	5	-3	22	2	14	3	-5
23	4	7	5	-2	23	2	15	3	-4
24	4	8	5	-1	24	2	16	3	-3
25	5	0	6	-11	25	2	17	3	-2
26	5	1	6	-10	26	2	18	3	-1
27	5	2	6	-9	27	3	0	4	-37
28	5	3	6	-8	28	3	1	4	-36
29	5	4	6	-7	29	3	2	4	-35
30	5	5	6	-6	30	3	3	4	-34
31	5	6	6	-5	31	3	4	4	-33
32	5	7	6	-4	32	3	5	4	-32
33	5	8	6	-3	33	3	6	4	-31
34	5	9	6	-2	34	3	7	4	-30
35	5	10	6	-1	35	3	8	4	-29
36	6	0	7	-13	36	3	9	4	-28
37	6	1	7	-12	37	3	10	4	-27
38	6	2	7	-11	38	3	11	4	-26
39	6	3	7	-10	39	3	12	4	-25
40	6	4	7	-9	40	3	13	4	-24
41	6	5	7	-8	41	3	14	4	-23
42	6	6	7	-7	42	3	15	4	-22
43	6	7	7	-6	43	3	16	4	-21
44	6	8	7	-5	44	3	17	4	-20
45	6	9	7	-4	45	3	18	4	-19

Reduced and Non Reduced Complicate Numbers

As told in the definitions what above presented suggest that, more in general, we can have 2 form of Complicate Numbers:

1- Reduced Form where the Rest is always bounded from 0 to $Rest_{max} \leq M_{n,i}$.

2- Non Reduced Form where the $Rest < P$.

As you understood I usually talk of the first type of Complicate Numbers

Analyzing the Rest

We know the Rest is not sufficient to define exactly who our (reduced) Complicate Number is, but in the most simple case it can define a Lower Bound. vice versa in another case we can be non able to do some evaluation, because for example we are working with Unknown Variables, that are just written as A, B, C or any other letter, just. In that case the complicate Modulus Algebra will give it best result allowing us to make some more deep concerning on the nature of such Rest, for example analyzing a Function that produce such rest (if possible). I will let this case for the next Volume since it is necessary, before enter in such type of Complicate Modulus Analysis, to know all the characteristic and behavior of the Complicate Modulus Numbers.

1) When is possible to find a Lower Bound:

Looking to the previous picture we can see that the Rest can be bigger than the Integer Root A.

If, for some reason, we have a Complicate Number (for a known power n) where the Rest is bigger than the n-th Power p^n of the Integer Root p, than we Reduce it simply returning it to be a classic Number P and then returning p to be:

$$p = \lfloor P \rfloor$$

and the Rest to be:

$$Rest = P - (\lfloor P \rfloor)^n = P - p^n$$

There is another case when, for some reason, we know just the Rest of such Complicate Number and we need to make an evaluation (for example in an inequality) of who can be the Minimum Integer Root of our Complicate Number's Rest.

With a simple concerning we can be sure that it cannot be a Number less than this value:

Calling p_{min} the Minimum Integer Root (given by the Rest we are talking of), it can be found considering that the Rest, for a Reduced Complicate Modulus Number, is always packed between the Last Gnomon represent the unknown Number P for what it's Root is:

$$p = \lfloor P \rfloor$$

and the Next One, so it is always:

$$M_{n,p} < Rest < M_{n,p+1}$$

Since:

$$M_{n,p} = (p^n - (p-1)^n)$$

Or :

$$(p^n - (p-1)^n) < Rest < ((p+1)^n - (p)^n)$$

So we have to solve the equation to have $X = p_{min}$:

$$(X^n - (X-1)^n) - Rest = 0$$

then we have to keep the Floor Value for X.
Here as example in the simplest case $n = 2$, so $M_{2,p_{min}} = 2p_{min} - 1$ so

$$p_{min} = \lfloor (Rest + 1)/2 \rfloor$$

Example1: $Rest = 40$, $n = 2$

$p_{min} = \lfloor ((40 + 1)/2) \rfloor = 20$

In fact the Last Gnomon is:

$M_{2,20} = 2 * 20 - 1 = 39$

And the Next one (is necessary to complete a Genuine Power of an Integer) is:

$M_{2,21} = 2 * 21 - 1 = 41$

So the Reduced Complicate Number having rest 39 can be just 20 since it is:

$20M_2 + 39 = 20^2 + 39 = 439$

1) When is NOT possible to find a Lower Bound:

Another case live the Mathematician at work with no results for several hundred years is when the Number is an Unknown Result of a function, so for example the result of the simple equation:

$$A^3 + B^3 =?C^3$$

where we ask also that $(A, B, C, n) \in \mathbb{N}^+$
So the question is: can be, for example, the Sum of two Cubic Powers, again the Cube of an Integer ?

$$5^3 + 6^3 =?7^3$$

So with our notation:

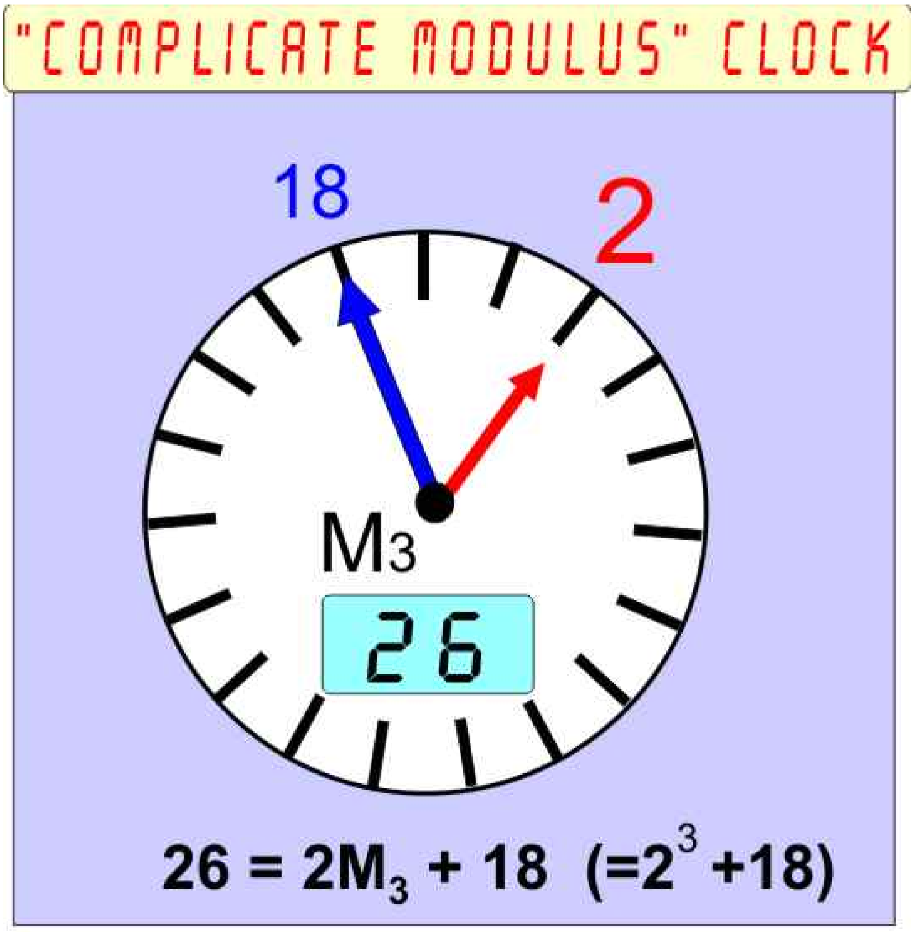
$$(5M_3 + 0) + (6M_3+) =?(7M_3 + 0)$$

This is Fermat Last Theorem, in the most simple non working case $n = 3$.

Fermat The Last Theorem ask the answer for any $(A, B, C, n) \in \mathbb{N}^+$.

The reply was one of the most hard problems of number Theory and has to wait several hundred years before Lord A. Whiles gives his final proof that just in case $n = 2$ we will have possible integer solutions. And the answer was given using Abstract Algebra and studying the behavior of some very sophisticated Modular curves (semi stable elliptic curves).

My Complicate Modulus Algebra was build to try find a most easy reply to this ancient problem on what will we return on in the next Vol.2. I would left away from such proof any abstract concept since humans (or, minimum, all as access to the simple math of Sum and Integrals) can understand why Fermat's trick doesn't works for $n > 2$. Lot must be investigated before we'll have in the hands all the weapons are necessary to rise our proof.



Here an example of how the 2 hands clock show $P = 26$ in terms of Integer Cubic Root (2) and Rest (18) It's possible to use it to show any $P \in N^*$ base any n-th power.

Here the clock runs, finally showing the number 26 modulus M_3:

It switch on and makes the first turn displaying 1, than it make another complete turn showings 8

But while running his 3th turn, it stops before concluding it:

The minutes hand stops onto the 18th division, as result of the recursive difference:

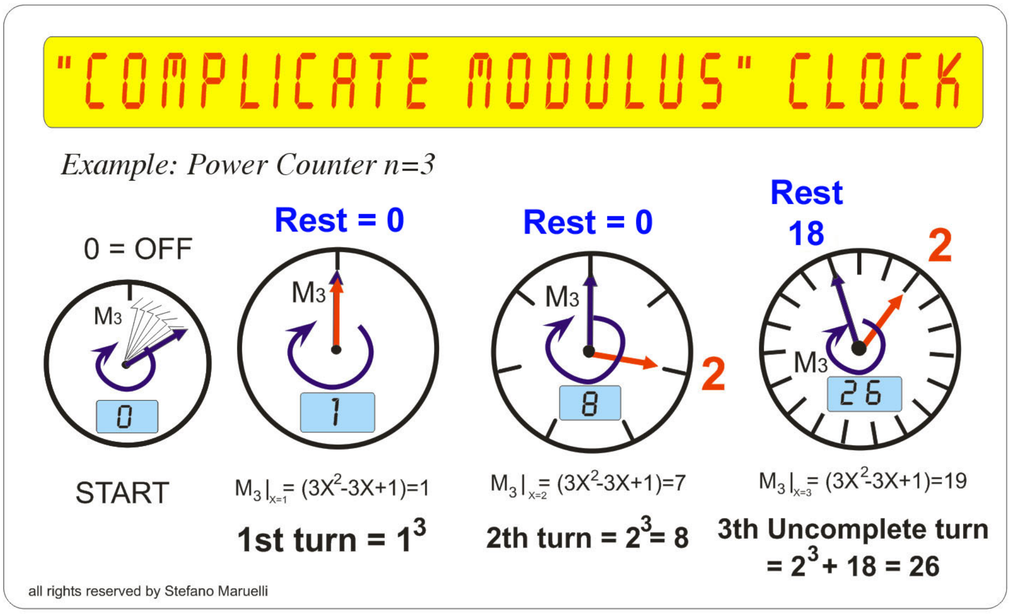
$26 - (3x^2 - 3x + 1)_{x=1} = 26 - 1 = 25$

$25 - (3x^2 - 3x + 1)_{x=2} = 25 - 7 = 18 = REST$

Since at the next turn we will have:

$$18 - (3x^2 - 3x + 1)_{x=3} = 18 - 19 = -1$$

So the clock stops at the previous 2nd turn, leaving us a REST=18



How the clock runs to show: 26 Modulus** M_3 **or $26^{1/3}$

Following this example it's possible to build-up all the n-th Clock we need to investigate if our number P is, or not the genuine Power of an Integer, we are searching for.

Note: from n=3 if we make a movie while running following root extraction of rising numbers P, we will see the shortest hour hand show us bigger and bigger Integer Roots, but due to the fact that the number of division of the clock rise each complete turn, it's angular movement, after it rise the maximum angular position of 2, it moves backwards to zero while P is rising. So the angular position of 3,4,5 etc... respect to the Zero, moves closer and closer to zero (the old 0-12 hours position on a classic known clock).

All that is a presentation of the first simple application of my Complicate Modulus Algebra. I spent 8 years of hard works to go deep enough in this simple trick, to let it becomes useful, so to let it work also with Rational, some Irrationals till infinitesimal P.

The Two-Hand-Clock with $n = 2$

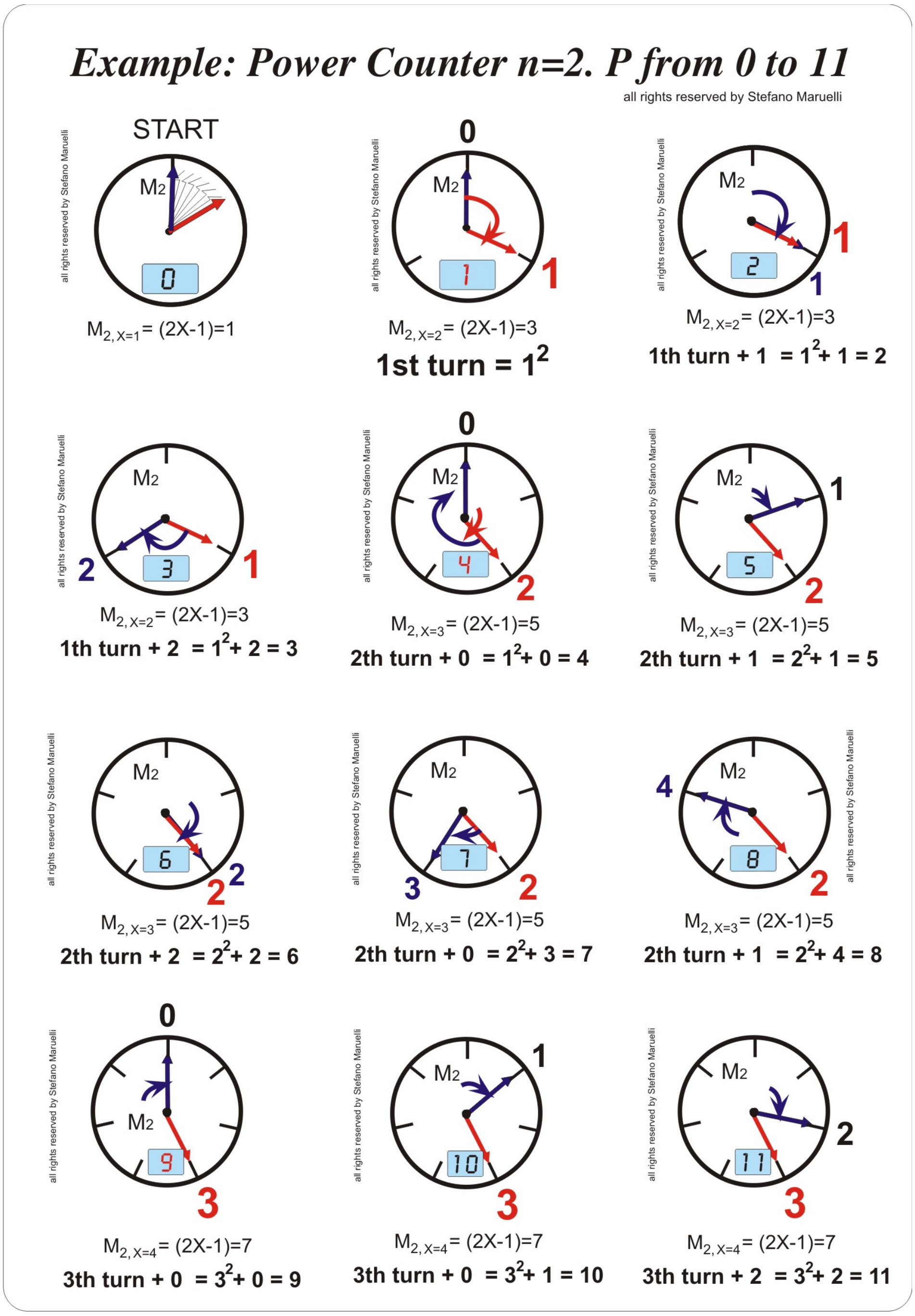


Figure 3: For $n = 2$ the movements of the two hands are clockwise and feel intuitive, while we will see it will be not from$n = 3$. The Hour's Hand will arrive at the limit for $P = \infty$ at a π position (6 o'clock).

The Two-Hand-Clock with $n = 3$

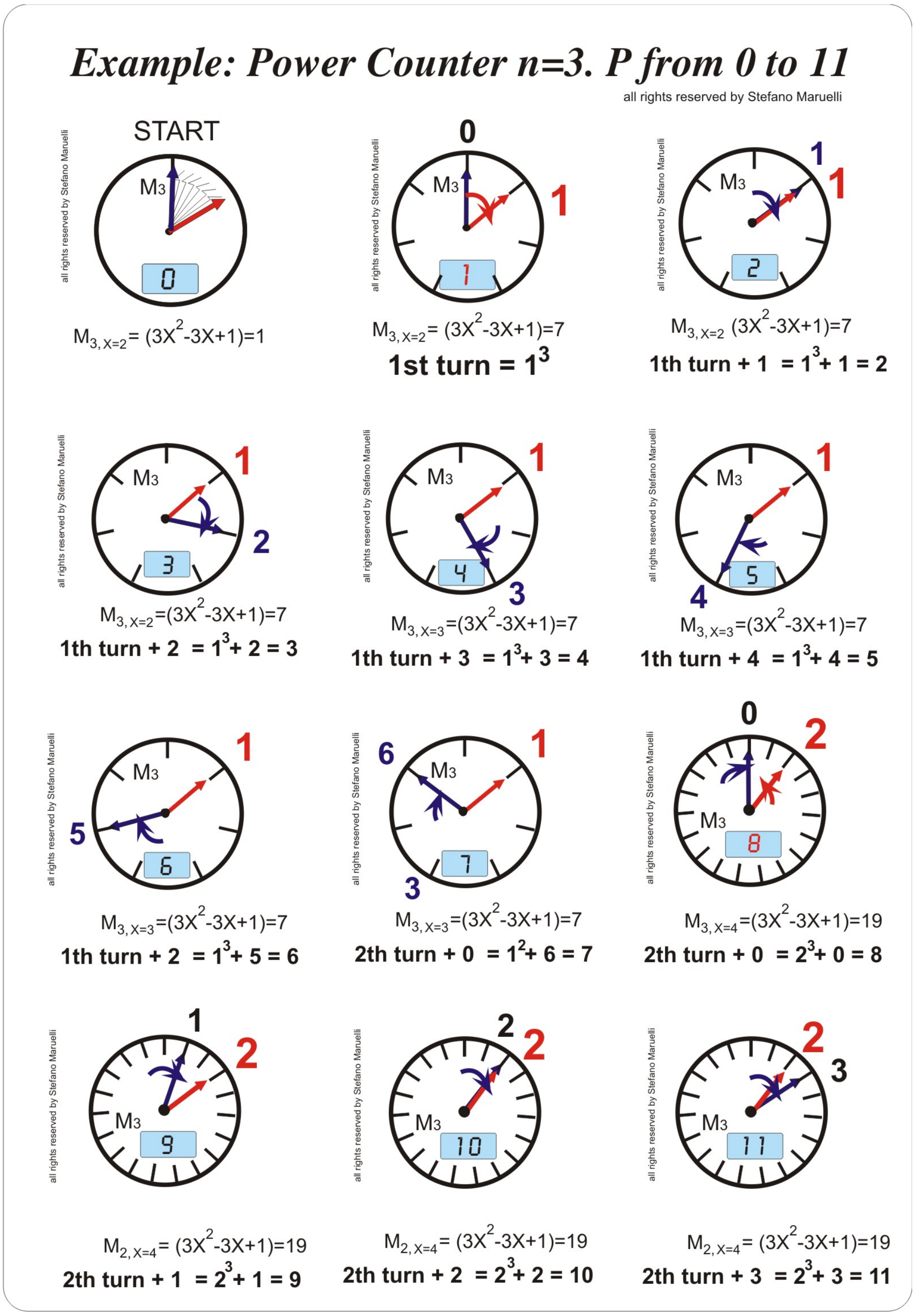


Figure 4: For $n = 3$, as you can see after reaching 7 the movements of the Rest's Hand is clockwise and feel intuitive while the movements of the Hour's Hand (Integer Root) is clockwise just till 1^n (and feel intuitive), while when it rise 2^n it start to moves counter clock wise since the number of divisions M_H rise faster that the new angle the hand has to do.

Hour's Hand Position in the case $n = 2$, and $n = 3$

P(x)	H	m	M_H	Y=ANGLE	P(x)	H	m	M_H	Y=ANGLE
1	1	0	1	360.00	1	1	0	1	360.00
2	1	1	3	120.00	2	1	1	7	51.43
3	1	2	3	120.00	3	1	2	7	51.43
4	2	0	5	144.00	4	1	3	7	51.43
5	2	1	5	144.00	5	1	4	7	51.43
6	2	2	5	144.00	6	1	5	7	51.43
7	2	3	5	144.00	7	1	6	7	51.43
8	2	4	5	144.00	8	2	0	19	37.89
9	3	0	7	154.29	9	2	1	19	37.89
10	3	1	7	154.29	10	2	2	19	37.89
11	3	2	7	154.29	11	2	3	19	37.89
12	3	3	7	154.29	12	2	4	19	37.89
13	3	4	7	154.29	13	2	5	19	37.89
14	3	5	7	154.29	14	2	6	19	37.89
15	3	6	7	154.29	15	2	7	19	37.89
16	4	0	9	160.00	16	2	8	19	37.89
17	4	1	9	160.00	17	2	9	19	37.89
18	4	2	9	160.00	18	2	10	19	37.89
19	4	3	9	160.00	19	2	11	19	37.89
20	4	4	9	160.00	20	2	12	19	37.89
21	4	5	9	160.00	21	2	13	19	37.89
22	4	6	9	160.00	22	2	14	19	37.89
23	4	7	9	160.00	23	2	15	19	37.89
24	4	8	9	160.00	24	2	16	19	37.89
25	5	0	11	163.64	25	2	17	19	37.89
26	5	1	11	163.64	26	2	18	19	37.89
27	5	2	11	163.64	27	3	0	37	29.19
28	5	3	11	163.64	28	3	1	37	29.19
29	5	4	11	163.64	29	3	2	37	29.19
30	5	5	11	163.64	30	3	3	37	29.19
31	5	6	11	163.64	31	3	4	37	29.19
32	5	7	11	163.64	32	3	5	37	29.19
33	5	8	11	163.64	33	3	6	37	29.19
34	5	9	11	163.64	34	3	7	37	29.19
35	5	10	11	163.64	35	3	8	37	29.19
36	6	0	13	166.15	36	3	9	37	29.19
37	6	1	13	166.15	37	3	10	37	29.19
38	6	2	13	166.15	38	3	11	37	29.19
39	6	3	13	166.15	39	3	12	37	29.19
40	6	4	13	166.15	40	3	13	37	29.19
41	6	5	13	166.15	41	3	14	37	29.19
42	6	6	13	166.15	42	3	15	37	29.19
43	6	7	13	166.15	43	3	16	37	29.19
44	6	8	13	166.15	44	3	17	37	29.19
45	6	9	13	166.15	45	3	18	37	29.19
46	6	10	13	166.15	46	3	19	37	29.19
47	6	11	13	166.15	47	3	20	37	29.19
48	6	12	13	166.15	48	3	21	37	29.19
49	7	0	15	168.00	49	3	22	37	29.19
50	7	1	15	168.00	50	3	23	37	29.19

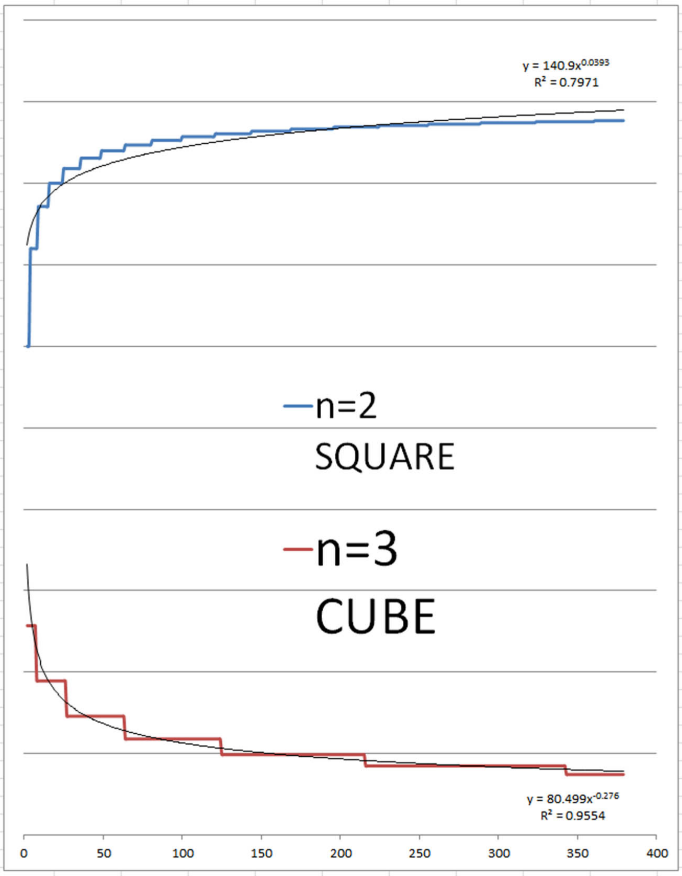


Figure 5: In the upper Blue graph the Position of the Hour's Hand in the case $n = 2$: the angle rise continuously and has 180 as limit. While in the case $n > 3$ the angle decrease (and has 0 as limit) due to the fastest increment of the number of the division for the Minute's Hand that will show the Rest.

Angle formula in the case $n = 2$:

Hour (Integer Root) $H = \lfloor (P)^{1/2} \rfloor$

minute (Rest) $m = P - (\lfloor (P)^{1/2} \rfloor)^2$

Number of Division at the current hour: $M_H = 2x - 1 = 2 * (H + 1) - 1$

$Angle = (360/M_H) * H$

Example: $n = 2$; $P = 15$

$H = \lfloor (P)^{1/2} \rfloor = \lfloor (15)^{1/2} \rfloor = 3$

$m = P - \lfloor (P)^{1/2} \rfloor = 15 - (\lfloor (P)^{1/2} \rfloor)^2 = 6$

$M_H = 2 * (H + 1) - 1 = 2 * (3 + 1) - 1 = 7$

$Angle = (360/M_H) * H = (360/7) * 3 = 154.29...$

Angle formula in the case $n = 3$:

Use the previous changing $n = 3$ and remembering that the new formula for M_H is:

$M_H = 3x^2 - 3x + 1 = 3 * (H + 1)^2 - 3 * (H + 1) + 1$

$Angle = (360/M_H) * H = ...$

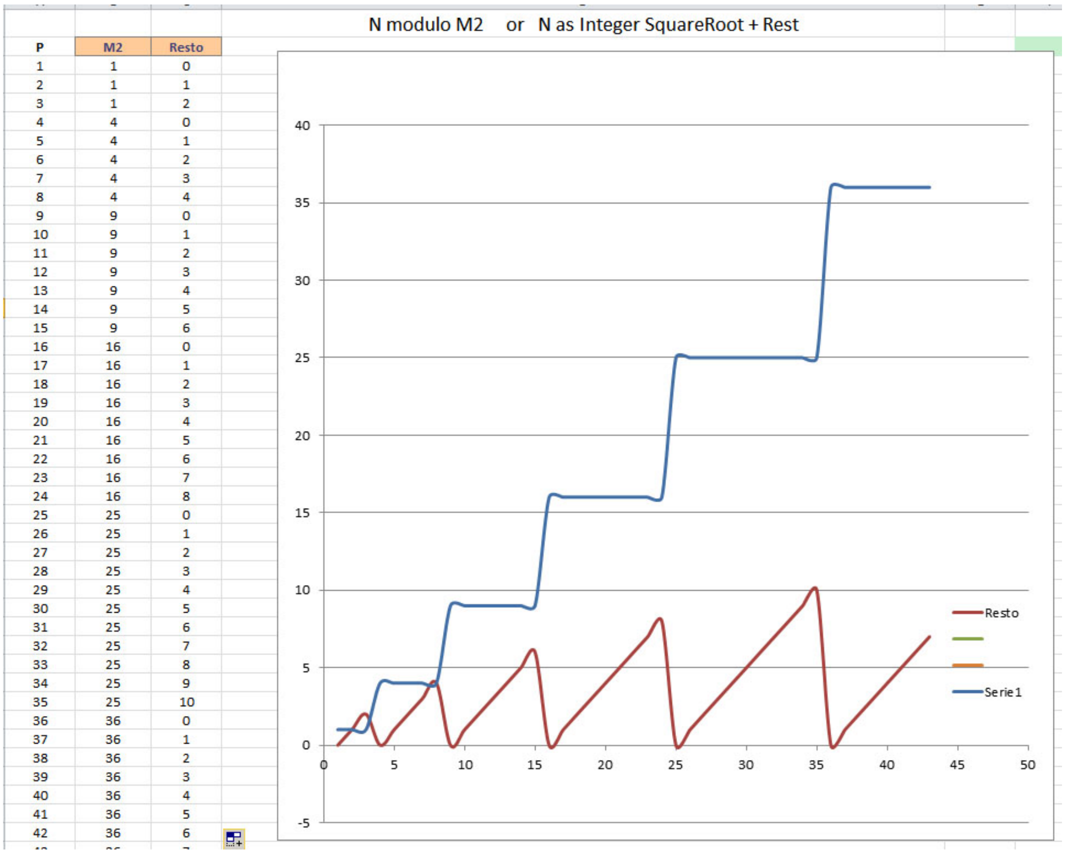
Of course we can now investigate in a more sophisticated clock where the Hours Hand moves continuously, so it means it always return, also alone, the exact value of the n-th root of P.

Unfortunately this doesn't help us to have an answer if $A^n + B^n = C^n$ has a solution for $C \in \mathbb{N}$.

A little Physics excursus:

I cannot resist to make a little Physics excursus, since

leaving the Square as Modulo, plus Rest this looks like the electron's jump mode, if we imagine that the Blue line is the actual orbit and the Red one the extra given energy. This explain why we must continuous to give energy to see a new jump, but that if we not give enough nothing change in the orbit.



P	M2	Resto
1	1	0
2	1	1
3	1	2
4	4	0
5	4	1
6	4	2
7	4	3
8	4	4
9	9	0
10	9	1
11	9	2
12	9	3
13	9	4
14	9	5
15	9	6
16	16	0
17	16	1
18	16	2
19	16	3
20	16	4
21	16	5
22	16	6
23	16	7
24	16	8
25	25	0
26	25	1
27	25	2
28	25	3
29	25	4
30	25	5
31	25	6
32	25	7
33	25	8
34	25	9
35	25	10
36	36	0
37	36	1
38	36	2
39	36	3
40	36	4
41	36	5
42	36	6

It's like a sort of an internal spring reaction: till we don't brake the spring it continuous to suck energy and nothing happen, but once it brakes, we see the jump. Of course there is nothing that brakes, what probably happen is that the electron such energy rising one parameter actually I don't know exactly how to call (twisting or else) that imply a jump just when rise his maximum possible value.

Nexus Number's Formula

Once the Integer derivative concept was understood, it is clear we can apply it recursively too, having the Second, the Third and the n-th derivative too.

The Nexus Numbers are know to be Numbers coming from a recursive difference. Here again the remainder to the ones coming from Squares and Cubes:

Integers	Power(s)	the Nexus Number(s)		
X	Y=X^2	Y'_i =X^2_(I+1)-X^2_I	Y''= Y'_(I+1)-Y'_I	Y'''= Y''_(I+1)-Y''_I
1	1	1	1	1
2	4	3	2	1
3	9	5	2	0
4	16	7	2	0
5	25	9	2	0
6	36	11	2	0
7	49	13	2	0
8	64	15	2	0
9	81	17	2	0
10	100	19	2	0

Integers	Power(s)	the Nexus Number(s)			
X	Y=X^3	Y'_i =X^3_(I+1)-X^3_I	Y''= Y'_(I+1)-Y'_I	Y'''= Y''_(I+1)-Y''_I	Y''''= Y'''_(I+1)-Y'''_I
1	1	1	1	1	1
2	8	7	6	5	4
3	27	19	12	6	1
4	64	37	18	6	0
5	125	61	24	6	0
6	216	91	30	6	0
7	343	127	36	6	0
8	512	169	42	6	0
9	729	217	48	6	0
10	1000	271	54	6	0

but will be now clear that they are also coming from an explicit general formula depends on the Binomial coefficients too, so on the Tartaglia's triangle:

Be the First Integer derivative equal to $Y'_I = M_n = (X^n - (X-1)^n)$ then we can define the second derivative of a parabolas of the type $Y = aX^n$ as:

$$Y''_I = Y''_I(X = i+1) - Y''_I(X = i)$$

that is quite boring to be calculated each time if one do not observe (and easily prove) that the result is:

$$Y''_I = Y'_I(X = i+1) - Y'_I(X = i) = (X^n - 2(X-1)^n + (X-2)^n)$$

and for so the 3rd Integer derivative will be:

$$Y'''_I = Y''_I(X = i+1) - Y''_I(X = i) = (X^n - 3(X-1)^n + 3(X-2)^n - (X-3)^n)$$

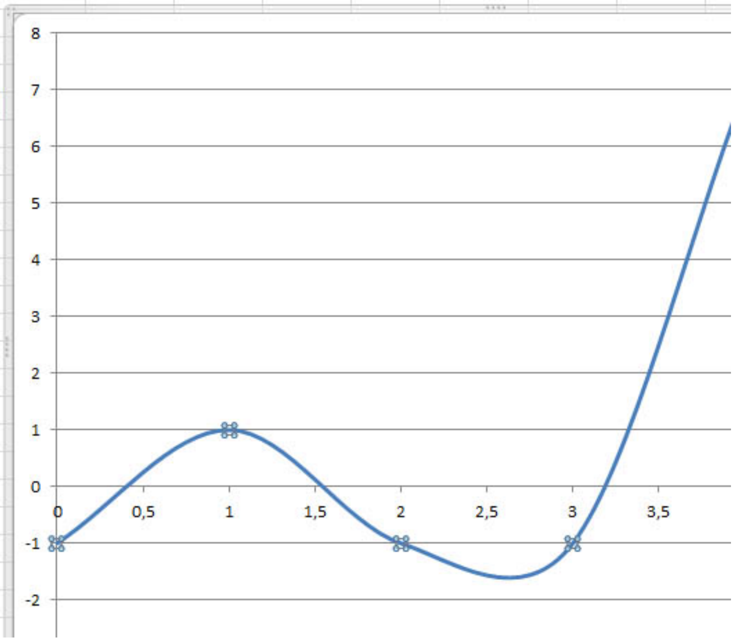
where the Triangular Structure know under many names one of the most famous in Italy is Tartaglia's coefficients (for $(X-1)^n$), emerges:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1
1 11 55 165 330 462 462 330 165 55 11 1

So we can apply such Derivative Concept to any Polynomial of the type: $Y = ax^n + bX^{n-1}$..... and of course also think what the imposition of $Y'_I = 0$ means in what I define the Exact Calculus, will produce here an apparently non so much interesting result is the abscissa of the second of two consecutive points having the same height.

But going ahead on my this paper I hope will rest clear in your mind that we can produce more interesting result using the Exact Calculus in the proper way, as we did for the known classic one.

	F(X)	Y'_I= D/DX(Fx)
X	X^3-5X^2+6X-1	3*X^2-(3+2*5)*X+(5+1+6)
0	-1	12
1	1	2
2	-1	-2
3	-1	0
4	7	8
5	29	22
6	71	42
7	139	68
8	239	100
9	377	138
10	559	182
11	791	232
12	1079	288
13	1429	350
14	1847	418
15	2339	492
16	2911	572
17	3569	658
18	4319	750
19	5167	848



Now keeping Known Calculus as example, we can search for the two Following Integer Ordinates having the same value, simply equating to zero the first (at this point) Integer derivative of an Integer coefficient polynomial:
Be: $Y = X^3 - 5X^2 + 6X - 1$ then the first integer derivative will be (applying the rule $Y' = a(X^n - (X-1)^n)$ each term):

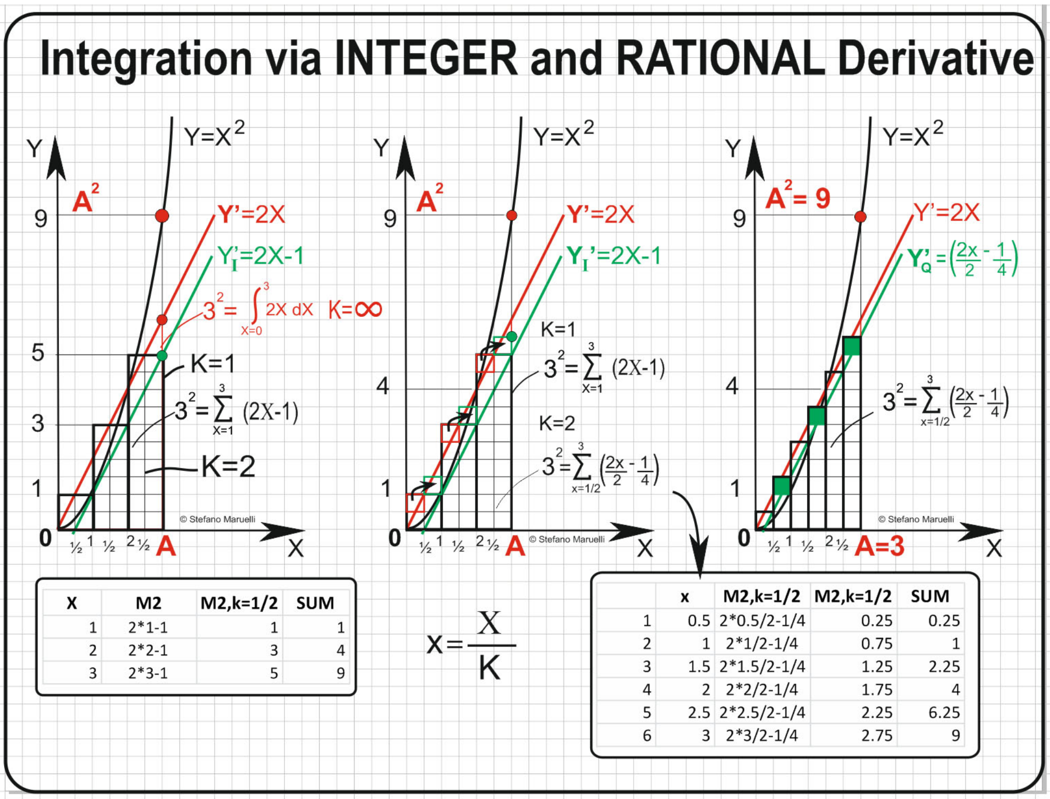
$$Y' = (X^3 - (X-1)^3 - 5*(X^2 - (X-1)^2) + 6 = 3X^3 - 3X + 1 - 5*(2X-1) + 6 = 3x^2 - 13x + 12$$

Then equating $Y' = 0$ we have:

$$3x^2 - 13x + 12 = 0; x_1 = 4/3; x_2 = 3$$

Since we are talking of Integer Points just, we keep just X=3 then we know now that for $X = 2$ and $X = 3$ the function has the same Ordinate $Y = -1$, means we have Two following points having the same Ordinate.

Chapt.6 STEP SUM: forcing the Sum operator to work with a Scaled Rational Index



Since we need to prove what in the previous chapters I've called the "Infinite Descent" trough the Convergent Series it's time to hack the Sum Operator. And this will be the MAIN POINT of All This WORK.

Is known it is possible to indicate bellow the Sum operator not just a rising Index i equal to an integer, but also an Index i selected by a Function that define just some Special Value that has to be Summed. Most common is the choice of Prime Number only, or just Odd or Even, etc... But always i was considered to be an Integer mute variable just.

And this was a Big Mistake because lie us blind for several hundred years !

what is known is that the $Step = 1$ is the integer difference between two following Integers having $Index = i$ and $i + 1$ for example, so it is:

$Step = (i + 1) - (i) = 1,$

But it's well known, for example, that is possible to Sum Odds just, or Primes just, or any other value defined by a pre-defined Function it will select Integer Index values $i \in \mathbb{N}$ just. So the value of the Step can be no longer equal to the Index is just an integer Number indicate where the pointer is, from Lower to Upper Limit.

Moreover it is possible, under conditions will follows, in the special case of Telescoping Sums (only!), so for example in case of Parables, so in the case the result o the Sum is $Y = X^n$. We will take for example the Sum:

$$A^2 = \sum_{X=1}^{A} (2X - 1); A \in \mathbb{N}^+$$

I start with the most simple case: $Step$ = Rational Step = $1/K$ with $K \in \mathbb{N}^+$

Remembering we want to hold the same result A^2 just unsinge a more fine Step Sum, having Step $1/K$, now we can introduce a Scale Factor $1/K^2$ and a factor K", so we divide all the terms of our Sum by K^2 , but remembering that we want to left unchanged the result, so for the known Sum's Rules we have to multiply the Upper Limit by K so we have:

$$A^2 = \frac{A^2}{K^2} * K^2 = \sum_{X=1}^{A*K} \left(\frac{2X}{K^2} - \frac{1}{K^2} \right)$$

Pls see the Appendix.1 to see the collection of Known Sum properties to refresh some of their properties, if necessary.

Now, pls be open mind and remember what is often done to solve some integral: we make an exchange of variable, so we can call: $x = X/K$, so changing X with $X = x * K$, if we respect the following conditions:

a) if and only if $a = A/K$ so if K is a Factor of A, so perfectly divide the Upper Limit A

- the Upper Limit A/K becomes: $(A/K) * K = A$ (with $K \in \mathbb{N}^+$)

- the Lower Limit $X = 1$ becomes: $x = 1/K$ (a Rational for now) so:

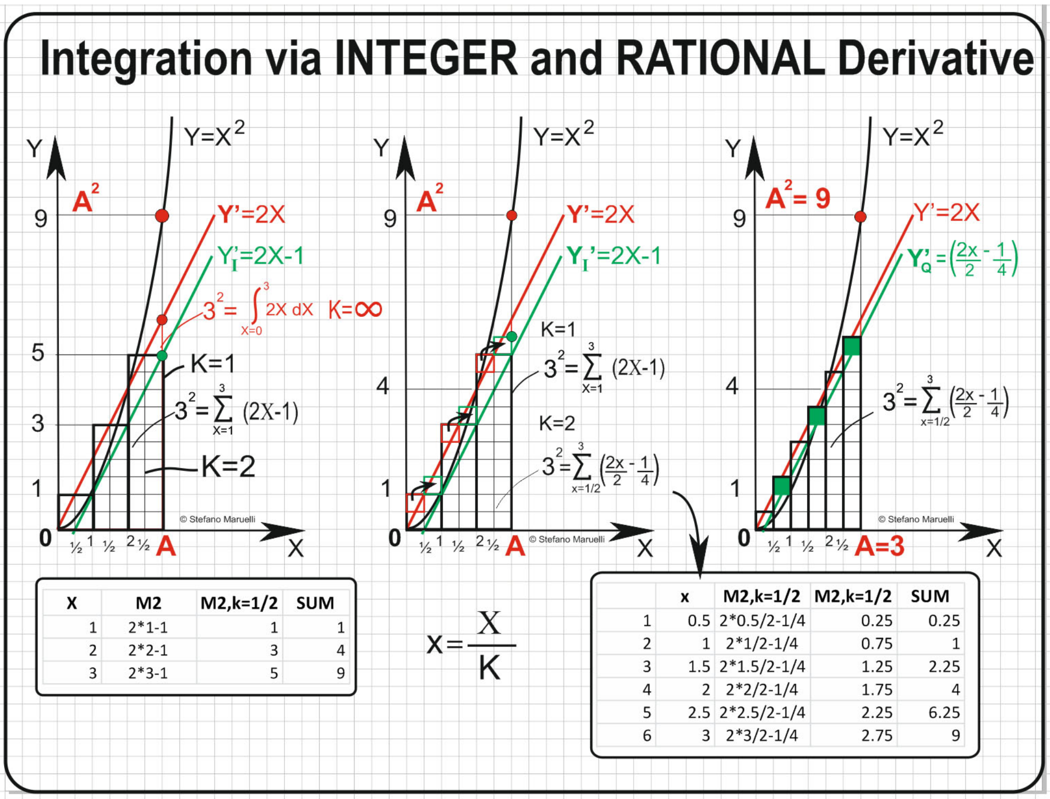
$$A^2 = \sum_{x=1/K}^{(A/K)*K} \left(\frac{2(x * K)}{K^2} - \frac{1}{K^2} \right)$$

Now we can simplify to have our new Step Sum, that moves of a quantity depends on the original Integer Index $i = 1, 2, 3, 4....$ but of a new scaled value we call Step $1/K$ from $1/K$ to A that is now allowed to be $A = P/K$ so $A \in \mathbb{Q}^+$, so the Index x will be $x = 1/K, 2/K, 3/K....A$:

$$A^2 = \sum_{x=1/K}^{A} \left(\frac{2x}{K} - \frac{1}{K^2} \right)$$

Where we start with: $A \in \mathbb{N}$ but can be now also $A \in \mathbb{Q}$

From this picture it's clear what happens: we are just scaling the abscissa (K=2 in that case), so we divide in 2 Each Base of Each Gnomon and, as consequence, we have to modify each Height respecting the rule: Missing Area equal to the Exceeding one.



I repeat it's intuitive for $n = 2$ since the Linear derivative $y = 2x$ helps us to see that for each Gnomon one Red Square $(1/2 * 1/2)$ has to jump right-up on the new, next, Gnomon we created (so from Red it becomes Green).

In this way we preserve the linear Rule of the derivative and of the Integer derivative, that the following Gnomon has to have an height that is 2 Units bigger than the previous one.

And we show also, that regardless which Unit we take: $1/2$ as shown here, $1/10$ or $1/10^m$ or in general $1/K^m$ with $K, m \in \mathbb{N}$ for the moment, we always perfectly square the derivative $y = 2x$ till a rational $A = P/K^m$

All this works also if we take as example the case n=3, in fact the term:

$M_n = [X^n - (X-1)^n]$ becomes: $M_3 = (3X^2 - 3X + 1)$

$$A^3 = \sum_{X=1}^{A} (3X^2 - 3X + 1)$$

Again now we can divide all the terms of our Sum by K^3 , remembering that if we want to left unchanged the result, we have to multiply the Upper limit by K so we have:

$$A^3 = \frac{A^3}{K^3} * K^3 = \sum_{X=1}^{A*K} \left(\frac{3X^2}{K^3} - \frac{3X}{K^3} + \frac{1}{K^3} \right)$$

Now we can call: $x = X/K$, so changing X with $X = x * K$, if we respect the following conditions:

a) if and only if K is a Factor of A, or perfectly divide A

- the Upper limit becomes: $(A/K) * K = A$ (with $K, A \in N^+$)

- the Lower Limit $X = 1$ becomes $x = 1/K$ so:

$$A^3 = \sum_{x=1/K}^{A} \left(\frac{3(x*K)^2}{K^3} - \frac{(x*K}{K^3} + \frac{1}{K^3} \right)$$

Now we can simplify to have our new Step Sum, that moves Step $1/K$ from $1/K$ to $A = P/K$, so the new Index x will be $x = 1/K, 2/K, 3/K....A$:

$$A^3 = \sum_{x=1/K}^{A} \left(\frac{3x^2}{K} - \frac{3x}{K^2} + \frac{1}{K^3} \right)$$

What is interesting is that K can be any Integer, but not just, as we will see in the next chapters.

Talking for the moment of Integer K is for so clear that it can be, for example, equal to : $K = k^m$

And this property will be of interest when I will show that we can also play with Rational value $k = 1/p$.

Example of a Step Sum step 1/10, for a Cube		
A=3	k=10	A^3 = 27
x	M3/k=3x^2/k-3x/k^2+1/k^3	SUM
0.1	0.001	0.001
0.2	0.007	0.008
0.3	0.019	0.027
0.4	0.037	0.064
0.5	0.061	0.125
0.6	0.091	0.216
0.7	0.127	0.343
0.8	0.169	0.512
0.9	0.217	0.729
1	0.271	1
1.1	0.331	1.331
1.2	0.397	1.728
1.3	0.469	2.197
1.4	0.547	2.744
1.5	0.631	3.375
1.6	0.721	4.096
1.7	0.817	4.913
1.8	0.919	5.832
1.9	1.027	6.859
2	1.141	8
2.1	1.261	9.261
2.2	1.387	10.648
2.3	1.519	12.167
2.4	1.657	13.824
2.5	1.801	15.625
2.6	1.951	17.576
2.7	2.107	19.683
2.8	2.269	21.952
2.9	2.437	24.389
3	2.611	27

In this Tab. the Cube of 3 calculated with a Step Sum, Step 1/10. From $x = 1/10$ to 3 we Sum 30 Gnomons $M_{3,K}$ calculate for each x.

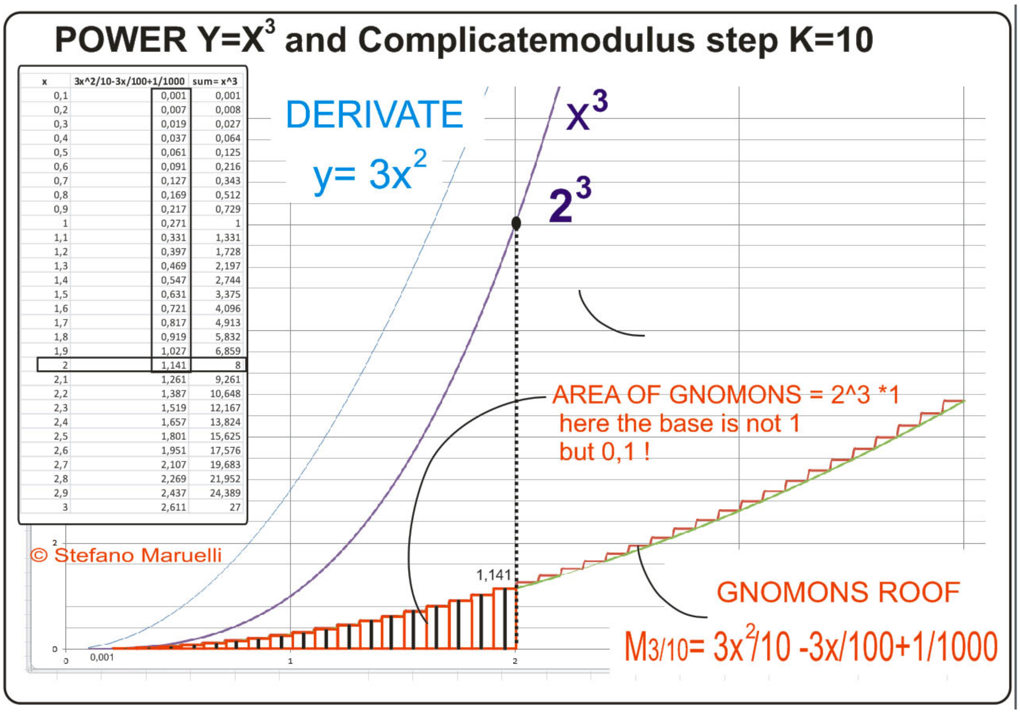
In the Appendix 2, you'll find also some interesting example of what happen to a Sum of such non conventional Terms when another function is applied to filter just few desired Rational.

Note: I know this new use of the Sums will be hard to be digested, but it will be more clear that it is not in conflict with the old notation and will let us show many interesting uses of this Step Sum that is now able to raise an Upper Rational limit $A = P/K$.

The General RATIONAL Complicate Modulus (or Gnomon's height function) $M_{n/K}$ for all $n-th$ power of Rational $A = P/K$ becomes:

$$M_{n,K} = \binom{n}{1}\frac{x^{n-1}}{K} - \binom{n}{2}\frac{x^{n-2}}{K^2} + \binom{n}{3}\frac{x^{n-3}}{K^3} - ... + / - \frac{1}{K^n}$$

So we have now a more useful instrument able to work with Rational, since I hope it's clear we can now stop to $0,5$ or $2,7$ or any other rational has 1 decimal digit only. If you need more digits, just rise K, if you need infinite just keep the right K divide the upper Rational limit P/K has infinite number of digits.



How to make the Cube of 2 with a Step Sum Step 0.1:

$2^3 = 8$ is equal to the sum of the Red Columns

Base = 0.1

and the **height** will be $M_{3/10}$

$M_{3,10} = 3x^2/10 - 3x/100 + 1/1000$

As we did for the Integers, we can now call the $y'_Q = M_{n,K}$ function: the **Rational derivative** that works for any $x \in Q$.

And to avoid any doubt I'm wrongly abusing of this notation, in the next chapter I'll show how to transform those two non continuous functions, into the well known continuous functions we call derivative / Integral.

Step Sum with Step $S > 1$:

I'll remember also how it's possible to manipulate the Sum without changing it's result in this way:

LEMMA 1: Step SUM with Step S>1

The condition to represent a Power of an Integer via Step Sum, where the index jumps Step $S > 1$ is:

Choose a Step S that perfectly divides the Upper limit A

So it's possible to write A^n using a Step Sum, Step: "S"$=$ factor of A with $S > 1$.

Example: we know how to write a square A^2

$$A^2 = \sum_{x=1}^{A} (2x - 1)$$

to hold the same result making for example just one step S, we need just to divide the number of Index, here is A, by A and multiply the Sum, or all the sum's terms by A:

$$A^2 = \sum_{x=1}^{A} (2x - 1) = \sum_{x=A}^{A} (2xA - A^2)$$

If $A = \pi_1 * \pi_2$ (where π_1 and π_2 two primes, or simply two factors of A) we can also write the Step Sum Step π_1 (or π_2) doing the same trick we did for the Rational Step Sum, this time Scaling UP that variables:

Again with an exchange of variable this time: $X = x * \pi_1$
the Lower Limit $X = 1$ becomes : $X = 1 * \pi_1 = \pi_1$
the Upper Limit $X = A$ becomes : $X = A * \pi_1/\pi_1 = A$
so the Sum becomes:

$$A^3 = \sum_{x=1}^{A} (3x^2 - 3x + 1) = \sum_{X=\pi_1}^{A} (3X^2 * \pi_1 - 3X * \pi_1^2 + 1 * \pi_1^3)$$

Here an example of a Step Sum, for Cubes are multiples of 3, Step 3:

x	M_{3 *3}	SUM
3	27	27
6	189	216
9	513	729
12	999	1728
15	1647	3375
18	2457	5832
21	3429	9261
24	4563	13824
27	5859	19683
30	7317	27000
33	8937	35937
36	10719	46656
39	12663	59319
42	14769	74088
45	17037	91125
48	19467	110592
51	22059	132651
54	24813	157464
57	27729	185193
60	30807	216000
63	34047	250047
66	37449	287496
69	41013	328509

Where:

$$A^3 = (3 * a)^3 = \sum_{x=1}^{A} (3x^2 - 3x + 1) = \sum_{X=3}^{A} (3X^2 * 3 - 3X * 3^2 + 1 * 3^3)$$

The minimum number of Step we can make is 1, keeping the variable equal to the Integer Root, but we can keep as $Step > 1$ any Integer factor of A, of one of their combinations.

From here we can immediately see that we are turning around the concept of factorization one can start to investigate.

Here an example of how many way we have to represent the Cube: 12^3 with a Step Sum having a Step>=1:

Table 4: How many way we have to represent the Cube: 12^3

Cube of 12 using STEP SUM, Step>=1

x	**X=1*x**	$3X^2*(1)-3X*(1^2)+1*(1^3)$	**SUM**	$SUM^{(1/3)}$
1	1	1	1	1
2	2	7	8	2
3	3	19	27	3
4	4	37	64	4
5	5	61	125	5
6	6	91	216	6
7	7	127	343	7
8	8	169	512	8
9	9	217	729	9
10	10	271	1000	10
11	11	331	1331	11
12	12	397	1728	12

x	**X=2x**	$3X^2*(2)-3X*(2^2)+1*(2^3)$	**SUM**	$SUM^{(1/3)}$
1	2	8	8	2
2	4	56	64	4
3	6	152	216	6
4	8	296	512	8
5	10	488	1000	10
6	12	728	1728	12

x	**X=3x**	$3X^2*(3)-3X*(3^2)+1*(3^3)$	**SUM**	$SUM^{(1/3)}$
1	3	27	27	3
2	6	189	216	6
3	9	513	729	9
4	12	999	1728	12

x	**X=4x**	$3X^2*(4)-3X*(4^2)+1*(4^3)$	**SUM**	$SUM^{(1/3)}$
1	4	64	64	4
2	8	448	512	8
3	12	1216	1728	12

x	**X=6x**	$3X^2*(6)-3X*(6^2)+1*(6^3)$	**SUM**	$SUM^{(1/3)}$
1	6	216	216	6
2	12	1512	1728	12

x	**X=12x**	$3X^2*(12)-3X*(12^2)+1*(12^3)$	**SUM**	$SUM^{(1/3)}$
1	12	1728	1728	12

The most General RATIONAL Complicate Modulus $M_{n,p/K}$:

Is well know we can Share an External (Integer for now) Factor into the Sum, so under certain conditions we can put it into the Step Sum, with an exchange of variable, in the same way. Then combining this Known Rule and the Exchange of variable previous one, we can Introduce into out Step Sum any Rational External Factor, adjusting Limits and Terms as will follow.
From the well known rule:

$$P * A^2 = \sum_{X=1}^{A} (2PX - P); (P, A) \in \mathbb{N}^+$$

We call P the External Factor and we start to consider the Special Case when $P = A$ so the External Factor is Equal to our Upper Limit, or equal to a Factor of the Upper Limit. In such case (If and only if !) we can make the Exchange of variable x= X*A:

$$A * A^2 = \sum_{X=1}^{A} (2AX - A) = \sum_{x=A}^{A^2} (2x - A)$$

at the condition that the Sum moves Step $=A$, but also due to the fact that Both Index and Terms, both behave linearly

For higher powers, f.ex. $n = 3$ the tricks works in the same way so there is a distribution of the External Factor into any term of the Sum:

$$A * A^3 = \sum_{X=1}^{A} (3AX^2 - 3AX + A)$$

But it is clear that we can NO LONGER make the exchange of variable $x = X * A$ due to the presence of higher degree terms (from X^2 on, for the higher power develop), and this still in the special case $P = A$.

To let the trick of the Exchange of Variable be possible we have to remember we can play with Irrational factors too, so looking to it as Cube (also in the case it is the Cube of an Irrational value too) we can properly share it into the Sum in this way:

$$(A^{1/3})^3 * A^3 = \sum_{X=1}^{A} (3(A^{1/3})^3 X^2 - 3(A^{1/3})^3 X + A)$$

and now putting: $x = A^{1/3} * X$ we can correctly operate the exchange of variable:

$$(A^{1/3})^3 * A^3 = \sum_{x=A^{1/3}}^{A*A^{1/3}} (3x^2 * (A^{1/3}) - 3x(A^{2/3}) + A)$$

This at the condition we move of the Irrational Step equal to $(A^{1/3})$.

Of course nothing change if instead of A we have any integer p, so most in general we can write:

$$(P^{1/3})^3 * A^3 = \sum_{x=P^{1/3}}^{A*P^{1/3}} (3x^2 * (P^{1/3}) - 3x(P^{2/3}) - P^{3/3})$$

So the sharing of the External Factor into the Step Sum looks exactly as the sharing of the divisor K we have seen in the previous chapter, so we are now ready to consider also a Rational External Factor p/K where $p = P^{1/3}$. Here an example of the result for $n = 3$:

$$\left(\left(\frac{P}{K}\right)^{1/3}\right)^3 * A^3 = \sum_{x=\left(\frac{P}{K}\right)^{1/3}}^{A*\left(\frac{P}{K}\right)^{1/3}} \left(3x^2 * \left(\frac{P}{K}\right)^{1/3} - 3x\left(\frac{P}{K}\right)^{2/3} - \left(\frac{P}{K}\right)^{3/3}\right)$$

So the most General RATIONAL Complicate Modulus (or Gnomon's height function) $M_{n,\frac{P}{K}}$ for all $n-th$ power of Rational $A = \frac{Q}{K}$ is:

$$P/K * A^n = \sum_{x=\left(\frac{P}{K}\right)^{1/n}}^{A*\left(\frac{P}{K}\right)^{1/n}} M_{n,\frac{P}{K}}; (P, Q, K, A) \in \mathbb{N}^+$$

where:

$$M_{n,\frac{P}{K}} = \binom{n}{1}(x^{n-1}) * \left(\frac{P}{K}\right)^{\frac{1}{n}} - \binom{n}{2}(x^{n-2}) * \left(\frac{P}{K}\right)^{\frac{2}{n}} + \binom{n}{3}(x^{n-3}) * \left(\frac{P}{K}\right)^{\frac{3}{n}} ... \pm \left(\frac{P}{K}\right)^{\frac{n}{n}}$$

In this way we can rise, for example any integer with the desired adjusted General RATIONAL Complicate Modulus.

For example we can rise the number $341 = 5^3 + 6^3$ showing it is not equal to the closest Cube 343, taking as example A=5, B=6 C=7 is a quasi solution of $A^3 + B^3 = C^3$:

Table 5: How to rise 341 Using a Cubic Irrational Modulus

X	$x = X * (341/343)$	$3x^2 * (341/343)^{1/3} - 3x * (341/343)^{2/3} + 341/343$	**Sum**
1	0,998052575	0,994169096	0,994169096
2	1,996105151	6,959183673	7,95335277
3	2,994157726	18,88921283	26,8425656
4	3,992210302	36,78425656	63,62682216
5	4,990262877	60,64431487	124,271137
6	5,988315452	90,46938776	214,7405248
7	6,986368028	126,2594752	341

Rule 11: Scaling the Sum. Index Vs. Terms Scaling / Shifting Rules

We see now the Last Set of Rules will help us to work with any problem involves Powers and Equalities:

A) - how to Scale (Up or down) the Upper Limit LEAVING THE RESULT UNCHANGED, so Rising/Lowering the Internal Terms of the SUM (JUST).

And, what happen trying to apply two modifications so:

B) - how to Scale (Up or down) the Upper Limit AND shifting the Lower one, LEAVING THE RESULT UNCHANGED, so Rising/Lowering the Internal Terms of the SUM (JUST), that is what Fermat state in his equation.

So in other terms for the Scaling Rule A:

A1) Is it possible, and under which conditions, to: Lower the UPPER LIMIT from A to $a < A$, just, leaving the result unchanged RISING the VALUE of the INTERNAL TERM/s ?

The answer is of course YES, with a trivial solution, if we introduce the Scaling Factor $\rho = (A/a)$:

$$\sum_{1}^{A} M_n = \sum_{1}^{a=A/\rho} \left(\frac{A}{a}\right)^n M_n = \sum_{1}^{a} \rho^n M_n = \rho^n \sum_{1}^{a} M_n$$

A2) Or, vice versa, is it possible, and under which conditions, to: Rise the LOWER LIMIT, for example from 1 to $LL > 1$, just LOWERING the VALUE of the INTERNAL TERM/s ?

The answer, for both case, is of course YES, with a trivial solution, if we introduce the Scaling Factor $\rho = (A/a)$:

$$\sum_{1}^{a} M_n = \sum_{1}^{A=a*\rho} \left(\frac{a}{A}\right)^n M_n = \sum_{1}^{A} (1/\rho)^n M_n = (1/\rho)^n \sum_{1}^{a} M_n$$

As we can see the Lowering Factor $\rho = (A/a)$ is of the same degree of the n-th Power we are working on, and is applied on all the terms of the Sum. The Factor can be, clearly, taken out from the Sum using the well known Sum's Rule. This will be clear in what follows once we introduce the exchange of Variable $x = X/K$ where I hope it's clear $K = \rho$

New Rule for scaling the Upper Limit of a Step Sum:

Since we can go Rational we can now make an operation will be useful, and it will be in the FLT proof: if we need to pass the Upper Limit of a Sum from A to B (with A and B integers), leaving the result of the Sum unchanged. From what we already know for going Rational so writing A^3 step $1/K$, and scaling the Upper Limit from A to B we have:

$$A^3 = \sum_1^A 3X^2 - 3X + 1 = \sum_{x=1/K}^{A} \frac{3x^2}{K} - \frac{3x}{K^2} + \frac{1}{K^3} = \sum_{x=A/B}^{B} \frac{3Ax^2}{B} - \frac{3xA^2}{B^2} + \frac{A^3}{B^3} \tag{1}$$

As we can see the Scaling of the limit needs the scaling of EACH term, to left the result of the Sum unchanged.

The proof is very simple once we put $\rho = B/A$ we have (remembering $M_n = (X^n - (X - 1)^n)$):

$$A^3 = \sum_1^A M_3 = \rho^3 \sum_1^B M_3 = (B/A)^3 \sum_1^B M_3 = \sum_1^B (B/A)^3(3X^2 - 3X + 1) \tag{1b}$$

then we take ρ^3 into the Sum and we apply the exchange of variable $X = \rho * x$ with $\rho = K = B/A$ to have:

$$A^3 = \sum_{x=1/K}^{B=A*\rho} \left(\frac{A}{B}\right)^3 \left(\frac{3x^2}{K} - \frac{3x}{K^2} + \frac{1}{K^3}\right) = \sum_{x=\frac{A}{B}}^{B=A*\frac{B}{A}} \frac{3A^3(x*B/A)^2}{B^3} - \frac{3A^3(x*B/A)}{B^3} + \frac{A^3}{B^3} \tag{1c}$$

that cancel out the B/A factors where possible becomes:

$$A^3 = \sum_1^A 3X^2 - 3X + 1 = \sum_{x=1/K}^{A} \frac{3x^2}{K} - \frac{3x}{K^2} + \frac{1}{K^3} = \sum_{x=A/B}^{B} \frac{3Ax^2}{B} - \frac{3xA^2}{B^2} + \frac{A^3}{B^3} \tag{1d}$$

And now an anticipation of the Vol.2: the reason of all this long work: Fermat ask himself if it is possible to perform a similar scaling, but working on the index dependent terms, only. F.ex for $n = 3$:

$$\sum_{x=1/K}^{A} \frac{3x^2}{K} - \frac{3x}{K^2} + \frac{1}{K^3} =? \sum_{X=1}^{C-B} (3(X+B)^2 - 3(X+B) + 1) \tag{2}$$

So the right scaling of a right hand is a Genuine Cube of A, so for the real equality becomes scaling the Upper Limit from (C-B) to A, (but has to holds the same internal terms !) taking $K = (C - B)/A$ that is for so:

$$\sum_{x=1/K}^{A} \frac{3x^2}{K} - \frac{3x}{K^2} + \frac{1}{K^3} = \sum_{x=(C-B)/A}^{A} \frac{3(C-B)x^2}{A} - \frac{3(C-B)^2x}{A^2} + \frac{(C-B)^3}{A^3} \tag{3}$$

While Fermat is asking this one :

$$\sum_{x=1/K}^{A} \frac{3x^2}{K} - \frac{3x}{K^2} + \frac{1}{K^3} =? \sum_{x=1/K}^{C-B} \frac{3(x+B)^2}{K} - \frac{3(x+B)}{K^2} + \frac{1}{K^3} \tag{4}$$

So if Fermat's Right hand of the (5) is right, so equal to A^3, it must be equal to the Right hand of the (3), while it is very different (and of course wrong):

$$\sum_{x=1/K}^{A} \frac{3x^2}{K} - \frac{3x}{K^2} + \frac{1}{K^3} =? \sum_{x=(C-B)/A}^{A} \frac{3(C-B)(x+B)^2}{A} - \frac{3(C-B)^2(x+B)}{A^2} + \frac{(C-B)^3}{A^3} \tag{5}$$

The proof will show that this is impossible in the Rational since it is necessary to go to the limit for $K \rightarrow \infty$ so with an integral to have the equality. So this means we have to rise to an Irrational limit (so one of the A,B, or C has to be an Irrational).

Here the table where you can see in numbers what happens:

- in the left Table we rise an Integer Upper Limit is $A = 5$ with a Step Sum with a generic rational step here is $1/K = 1/5$. The result is always A^3 since for Integers this sum is K invariant.

- While on the right top we see how to rise the same A^3 value, but with a rational Sum from 1 to 6, so having 6 step. The Rule for this scaling is the one in the previous page.

- in the last one (5), viceversa we will see what happens into the right hand of a Fermat Equation $A^3 = C^3 - B^3$ once cutted and shifted to the origin in the quasi triplet case A=5, B=6, C=7: applying the right scaling Rule for the Upper Limit, so as we did for A to B, but from $(C - B)$ to A we see we cannot rise the genine power of an integer is 125, since we clearly have, compared with the (3), same limits, but bigger Terms.

The aim here is just to show an example of this Rules, for the Proof you've to wait my Vol.2 since it comes applying a limit and you probably need to learn what follows, so the Maruelli's Integral via limit of this Step Sum. It looks like, but is not the same of Riemann one.

FLT-N3-LEFT-HAND as GENUINE CUBE of A via STEP SUM K=A			
A=5		Right Modulus for CUBES	GENUINE CUBE of x
X	x=X/A	3x^2/A-3x/A^2+1/A^3	SUM
1	0.2	0.008	0.008
2	0.4	0.056	0.064
3	0.6	0.152	0.216
4	0.8	0.296	0.512
5	1	0.488	1
6	1.2	0.728	1.728
7	1.4	1.016	2.744
8	1.6	1.352	4.096
9	1.8	1.736	5.832
10	2	2.168	8
11	2.2	2.648	10.648
12	2.4	3.176	13.824
13	2.6	3.752	17.576
14	2.8	4.376	21.952
15	3	5.048	27
16	3.2	5.768	32.768
17	3.4	6.536	39.304
18	3.6	7.352	46.656
19	3.8	8.216	54.872
20	4	9.128	64
21	4.2	10.088	74.088
22	4.4	11.096	85.184
23	4.6	12.152	97.336
24	4.8	13.256	110.592
25	5	14.408	125

Maruelli-FLT-N3-LEFT-HAND as SHIFTED a GeNUINE CUBE of A			
B=6	A=5	GENUINE "CUBE of A", WITH B=6 AS NEW UPPER LIMIT	
X	x=A*X/B	3Ax^2/B-3A^2*x/B^2+A^3/B^3	SUM
1	0.83333333	0.578703704	0.5787037
2	1.66666667	4.050925926	4.62962963
3	2.5	10.99537037	15.625
4	3.33333333	21.41203704	37.037037
5	4.16666667	35.30092593	72.337963
6	5	52.66203704	125

Right Hand A=5 B=5 C=7 "quasi triplet" example (classic Sum)			
A=5	C-B=1	NON GENUINE "CUBE of A", FOLLOWING FLT REQUEST	
X		3(X+B)^2-3(X+B)+1	SUM
1		127	127

Right Hand A=5 B=5 C=7 "quasi triplet" example (Step Sum)			
A=5	C-B=1	NON GENUINE "CUBE of A", Rational Step K=(A/(C-B))	
X	x=X*(C-B)/A	3(C-B)(x+B)^2/A-3(C-B)^2*(x+B)/A^2+(C-B)^3/A^3	SUM
1	0.2	22.328	22.328
2	0.4	23.816	46.144
3	0.6	25.352	71.496
4	0.8	26.936	98.432
5	1	28.568	127

Chapt.7: From Step Sum to the Integral:

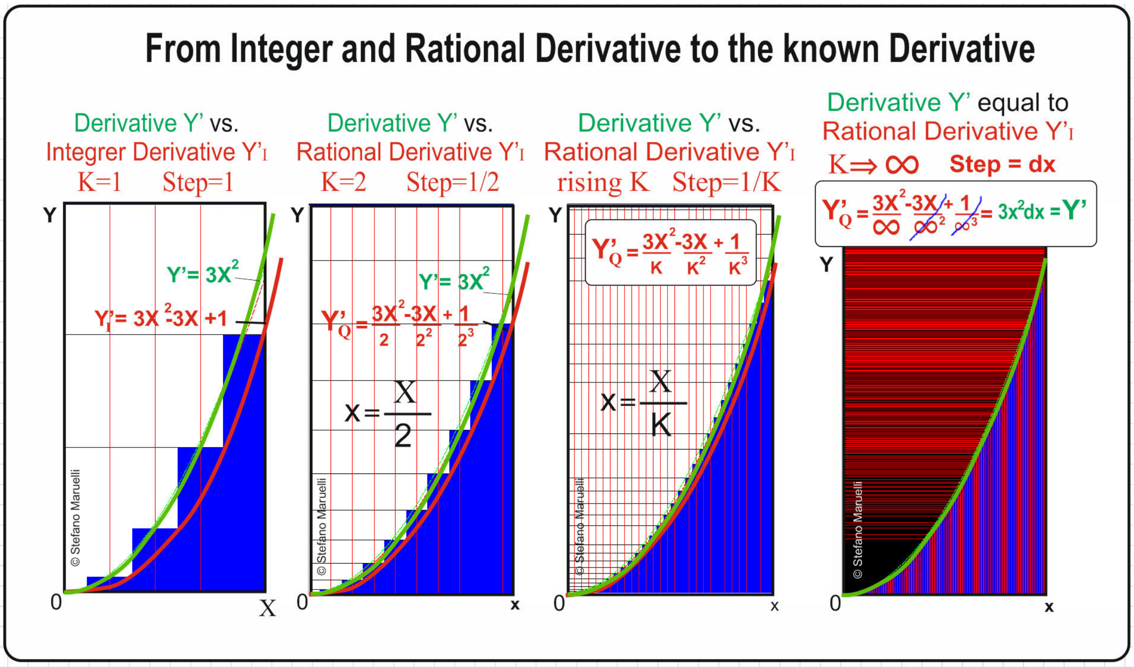
We enter now in the most interesting part of the Rational Calculus, what is known as the Finite Difference Analysis, passing from the Sum, to the Step Sum, to the Limit, showing that the Telescoping Sum property lead to the Integral, but in an interesting way:

we have no more, as in the Classic Riemann Integral an approximation of via via more close Areas, approximation, since talking of derivative of Parabolas we know we have an invariant: so don't care if we square the derivative with our Gnomons, or rational Gnomons, or via Integral: we always get the same value (under few simple conditions).

I'll aslo show that in a very similar way we can Bound some Irrationals between a Lower and an Upper Integer and then Rational Limits that becomes our Irrational Value just once we push the divisor $K \to \infty$.

From Step Sum to the Integral:

If we keep for example the Complicate Rational Modulus for Cubes: $M_{3,K} = \frac{3x^2}{K} - \frac{3x}{K^2} + \frac{1}{K^3}$ and we fix, for example (since it is true for any $K \in \mathbb{N}^+ : K = 10^m$ then pushing $m \to \infty$ we have back the well know integral, as shown in this picture:



The telescoping Sum Properties assure us that Power's of Integers, so all the derivative of $Y = X^n$, can be perfectly squared with columns of any BASE from 1, but as seen due to the fact that we can scale any picture as we want, we can also think to increase the number of Gnomons keeping a littlest base $1/K$ instead of 1 (or more under certain conditions), but we can also push $K \to \infty$ to move Step dx, so having back an integral.

Starting from $A \in N^+$ we can write A^n as a Sum, or as a Step Sum or, at the Limit as Integral remembering the exchange of variable $x = X/K$ in each X dependent Term (in this way we cut by K^n the Sum of the terms), and in the Lower Limit (and in this way we multiply by K the number of index, what I call the Step balancing the reduction of the Terms, as shown in the first chapters), having:

$$A^n = \sum_{X=1}^{A} M_n = \sum_{x=\frac{1}{K}}^{A} M_{n,K} = \lim_{K\to\infty} \sum_{x=\frac{1}{K}}^{A} M_{n,K} = \int_0^A nx^{(n-1)}dx$$

Example for $n = 3$, putting $x = X/K$:

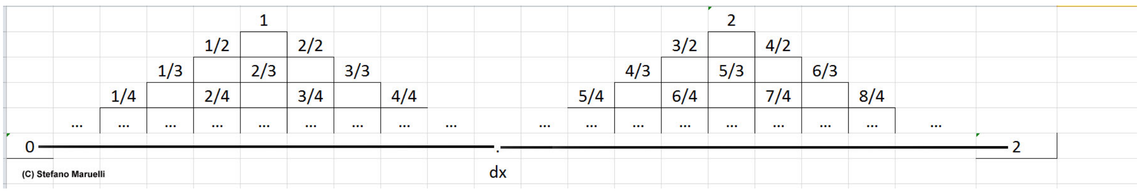
$$A^3 = \sum_{X=1}^{A}(3X^2 - 3X + 1) = \sum_{x=1/k}^{A}\left(\frac{3x^2}{K} - \frac{3x}{K^2} + \frac{1}{K^3}\right)$$

Or:

$$A^3 = \lim_{K\to\infty} \sum_{x=1/K}^{A}\left(\frac{3x^2}{K} - \frac{3x}{K^2} + \frac{1}{K^3}\right) = \int_0^A 3x^2dx = A^3$$

It's easy to prove this Limit with the classic technique, but also note that we have a proof of the Transcendental Law of Homogeneity for $K \to \infty$ that, in this case: $3x/K^2$ and $1/K^3$ are vanishing quantities (are infinitesimal of bigger order) respect to the First Term $3x^2/K$ since it depends just on $f(x)/K$, that is our non vanishing quantity dx

Just to remember how Numbers are organized:



We can therefore state out what was one of my first Theorem here (discovered several years before the previous Rule):

If we are working with a power of an Integer, only, the result of the Sum / Step Sum is independent from the K we choose:

If $A \in N^*$ than we can write A^n as:

$$A^n = \sum_{x=1}^{A} M_n = \sum_{x=1/K}^{A} M_{n,K} = \lim_{K\to\infty} \sum_{x=1/K}^{A} M_{n,K} = \int_0^A 3x^2 dx$$

Remembering that:

$$M_{n,K} = \binom{n}{1}\frac{x^{n-1}}{K} - \binom{n}{2}\frac{x^{n-2}}{K^2} + \binom{n}{3}\frac{x^{n-3}}{K^3} + ... + / - \frac{1}{K^n}$$

If $A \in Q : A = P/K$ than we can write A^n just as:

$$A^n = \sum_{x=1/K}^{A} M_{n,K} = \lim_{K\to\infty} \sum_{x=1/K}^{A} M_{n,K} = \int_0^A 3x^2 dx$$

If $A \in R$ with A= Irrational, than we can, in general, write A^n as:

$$A^n = \lim_{K\to\infty} \sum_{x=1/K}^{A} M_{n,K} = \int_0^A 3x^2 dx$$

But now the interesting concerning: if the Irrationality of A depends on a known factor, for example $A = \sqrt{2} * a$ where $a = r/s \in (Q)$,

than is possible to hack again the Step Sum, fixing an Irrational Step $S = 1/K * \sqrt{2}$ to let the Step Sum works with a Finite Number of Irrational Steps.

I know this left some non expert reader a little stuck, but after few minutes of check you will se it works and this will be very useful once we will look into Fermat's Last Theorem and Beal conjecture.

Proof in the most simple case $n = 2$:

Given: $a, k \in N^+$ we can write:

$$\sum_{x=1}^{ak}\left(\frac{2x}{k^2}-\frac{1}{k^2}\right) = a^2 = \int_0^a 2x\ dx.$$

Proof:

$$\sum_{x=1}^{ak}\left(\frac{2x}{k^2}-\frac{1}{k^2}\right) = \frac{1}{k^2} * \left\{\left(2 * \sum_{x=1}^{ak} x\right) - \sum_{x=1}^{ak} 1\right\} =$$

$$= \frac{1}{k^2} * \{(ak)(ak) + \cancel{ak} - \cancel{ak}\} = \frac{1}{\cancel{k^2}} * (a^2\cancel{k^2}) = a^2 = \int_0^a 2x\ dx$$

For $n > 2$ it follows in the same way (just with more vanishing Terms). But to understand the fact that there is continuity between the Integer Sum and the Integral, also the Old Mathematician has to digest that:

- The Mute Property of the Index was a False Math Mito, if taken in the sense that it has nothing to tell to us, in fact,

as shown, we can make a change of variable calling $x = X/K$ so we have a K times scaled variable and to Left unchanged the result we can write:

$$A^2 = \sum_{x=1}^{AK}\left(\frac{2x}{K^2}-\frac{1}{K^2}\right) = \sum_{x=1/K}^{A}\left(\frac{2x}{K}-\frac{1}{K^2}\right) =$$

$$= \lim_{K\to\infty} \sum_{x=1/K}^{A}\left(\frac{2x}{K}-\frac{1}{K^2}\right) = \int_0^A 2x\ dx$$

The proof it's immediately given once will be clear that :

- For the Lower Limit, the first $1/K$ step when $K \to \infty$ becomes $1/K = 0$

- $1/K^2$ it's an infinitesimal of Bigger Order respect to $1/K$, than it vanishes.

- The first term divisor $1/K$ becomes in the standard notation: $1/K = dx$ for $K \to \infty$

Note: Is very important to remember that the Integral of the Rational derivative, once it is considered as a Continuous Curve always satisfy this property:

$$\int_0^A M_{n,x}\ dx < \int_0^A nX^n dx$$

In fact is clear that, for example for $n = 3$:

$$\int_0^A (3x^2 - 3x + 1)dx < \int_0^A 3x^2 dx$$

And will be also important for what I'll show in the next chapters to note that in case we take as Complicate Modulus:

$M_{n,x+1} = (X+1)^n - X^n)$ we have (Except for A=1)

$$\int_0^A M_{n,x+1}\ dx > \int_0^A nX^n dx$$

for example for $n = 3$:

$$\int_0^A (3x^2 + 3x - 1)dx > \int_0^A 3x^2 dx$$

And. of course this will holds true also if we go Rational:

$$\int_0^A M_{n,K,x}\ dx < \int_0^A nX^n dx$$

In fact is clear that, for example for $n = 3$, for any $K \in \mathbb{Q}^+$:

$$\int_0^A \left(\frac{3x^2}{K} - \frac{3x}{K^2} + \frac{1}{K^3}\right) dx < \int_0^A 3x^2 dx$$

And in case we take as Rational Complicate Modulus the next step $M_{n,K,x+1}$ we have (Except for A=1) for example for $n = 3$:

$$\int_0^A M_{n,K,x+1}\ dx > \int_0^A nX^n dx$$

$$\int_0^A \left(\frac{3x^2}{K} + \frac{3x}{K^2} - \frac{1}{K^3}\right) dx > \int_0^A 3x^2 dx$$

The concept of Bound will become very useful more ahead in this work once we will work on Fermat the Last Theorem and on the Irrational numbers. Will be shown in the next pages how the approach to the Limit behave rising $1/K$:

Differently from how the Riemann Sum smoothly approach the Riemann Integral, here we will see that the approach to the Limit is convergent (in media) but defined by a non smooth function. For us is enough to prove it is convergent in Media and that both the Best Approximated Points of the Lower and of the Upper Bound are converging too. So this convergence is not as intuitive as the Riemann one is. In the next chapters a complete explanation of this last concerning.

Chapt.8: Extraction of the Rational n-th root, with a fixed number of digit precision

We saw in Chapt.3 that is possible to make the n-th root of an Integer using the Recursive Difference δ method.

After showing how the Step Sum works, so how to rise a Power of a Rational using a Step Sum, Step $1/K$, I'll show now how to make the Rational n-th Root p of a Rational $P \in \mathbb{Q}$. I start keeping $P \in \mathbb{N}$ since at the end of the first example will be clear how to go over in the Rational too.

We know that regardless if $A = Q/K \in \mathbb{Q}$ or $A \in \mathbb{N}$ we can write, for example for $n = 3$:

$$A^3 = \sum_{x=1/K}^{A} \left(\frac{3x^2}{K} - \frac{3x}{K^2} + \frac{1}{K^3} \right)$$

So is clear that if we have a number $P = 31$, of what we want to know, for example, the cubic root with 1 digit approximation, we have to repeat backward the step we made to have it's cube keeping $K = 10$ and the Complicate Modulus M_3. More in general to have m digits we have to choose $K = 10^m$, since we are working with Decimal Base Numbers.

In the next table I've plotted:

$x = RationalRoot$; $M_{3,K=10} = (3x^2/10 - 3x/10^2 + 1/10^3)$; Partial Sum ; Difference: 31 - Partial Sum

In case P is not a Perfect Cube, we have at the end of the recursive Difference a $Rest$ that is littlest than the next Rational (K=10) Rational Cubic Gnomon so $M_{3/10}$ calculated at $x = p + 1$ (where I remember p is always in our notation the Integer Cubic Root of P).

If rising K it's possible to arrive at $Rest = 0$ then P has a Rational root, vice versa, as I said in Chapt.3, this case will be divided in two:

1- it has an infinite, Periodic, number of Decimal Digits so it's a Rational, or

2- it has an infinite, Non Periodic, number of Decimal Digits so it's an Irrational.

In the case of 31 we have the suspect it has an Irrational Root since rising K, the Rest becomes smaller and smaller, but we need to push K to the limit $K \to \infty$ to see the Rest vanishing, and we can't recognize a Period in that number.

Using this Algorithm we can therefore prove whether a number P has or not an Integer Root (so is, or not a Perfect h-th Power), and since this method always gives the first right $m - th$ digits choosing $k = 10^m$, than it always gives an Approximated Lower Value of the Root.

In case the Rational Root has a Period bigger than our computational power, and we are not able to discover with other method is there is a factor $K = \pi_1$ that leads to a Zero Rest once we set a Recursive Step Difference $K = \pi_1$, it is clear we cannot distinguish between a Rational or an Irrational.

Is also clear this is an easy / non fast, non computational useful method, but I'll show that this will be a very useful method to solve Power Problems, like prove the Irrationality of a Number, prove Fermat the Last, Beal Conjecture etc. Here I patch the table of the Recursive Difference $K = 10$ that show us the Cubic Root of 31 with one Decimal digit. Than I'll do the same rising $K = 100$

It must be clear that this is not the 'Infinite Descent' we know it was used by Fermat's and Newton's use for their proof for $n = 4$ and $n = 3$, but is a new more powerful method.

	K=10		P=31
x	$M_{3,10,x}$ = 3x^2/K-3x/K^2+1/K^3	SUM	Difference
0,1	0,001	0,001	30,999
0,2	0,007	0,008	30,992
0,3	0,019	0,027	30,973
0,4	0,037	0,064	30,936
0,5	0,061	0,125	30,875
0,6	0,091	0,216	30,784
0,7	0,127	0,343	30,657
0,8	0,169	0,512	30,488
0,9	0,217	0,729	30,271
1	0,271	1	30
1,1	0,331	1,331	29,669
1,2	0,397	1,728	29,272
1,3	0,469	2,197	28,803
1,4	0,547	2,744	28,256
1,5	0,631	3,375	27,625
1,6	0,721	4,096	26,904
1,7	0,817	4,913	26,087
1,8	0,919	5,832	25,168
1,9	1,027	6,859	24,141
2	1,141	8	23
2,1	1,261	9,261	21,739
2,2	1,387	10,648	20,352
2,3	1,519	12,167	18,833
2,4	1,657	13,824	17,176
2,5	1,801	15,625	15,375
2,6	1,951	17,576	13,424
2,7	2,107	19,683	11,317
2,8	2,269	21,952	9,048
2,9	2,437	24,389	6,611
3	2,611	27	4
p= 3,1	**2,791**	**29,791**	**1,209**
3,2 too big	2,977	32,768	-1,768

To make the cubic root of 31, with 1 digit precision we have to keep $K = 10$, then use the Gnomon: $M_{3/10} = 3x^2/10 - 3x/10^2 + 1/10^3$. Starting from 0.1 going ahead till the Next Step it gives a Gnomon that is Too Big to be subtracted from the Rest we have.

I'll show now what happens rising $K = 10^2$: a new right digit will appear, this means this algorithm alway gives us the approximation by defect of the true root till the $m - th$ digit, here now $m = 2$.

K=100			P=31
$M_{3,100,x}$ = 3x^2/100-3x/100^2+1/100^3			
x	$M_{3,100,x}$	SUM	Diff. 31-$M_{3,100,x}$
0.01	0.000001	0.000001	30.999999
0.02	0.000007	0.000007	30.999992
0.03	0.000019	0.000019	30.999973
0.04	0.000037	0.000037	30.999936
0.05	0.000061	0.000061	30.999875
0.06	0.000091	0.000091	30.999784
0.07	0.000127	0.000127	30.999657
0.08	0.000169	0.000169	30.999488
0.09	0.000217	0.000217	30.999271
0.1	0.000271	0.000271	30.999
..	..	..	..
3.07	0.281827	0.281827	2.065557
3.08	0.283669	0.283669	1.781888
3.09	0.285517	0.285517	1.496371
3.1	0.287371	0.287371	1.209
3.11	0.289231	0.289231	0.919769
3.12	0.291097	0.291097	0.628672
3.13	0.292969	0.292969	0.335703
p= 3.14	0.294847	0.294847	0.040856
3.15 too big	0.296731	0.296731	-0.255875

So if you need m digits, you've to take $K = 10^m$

In case you rise the last significant digit, and you continuous to rise K, you'll just find other digits equal to zero.

If you see a periodic sequence of results, then you are sure you've a Rational, but if no zeros nor periodic sequence is shown, than, if your are not able to discover the right Factor K, of $P = Q/K \in \mathbb{Q}$, you still lie in the doubt if you are playing with a Rational or an Irrational. To prove you are playing with a Rational This is another proof that Factorization play a very important rule in several Math problems.

I choose Numbers $Base = 10$, but we can did the same for any $Base$.

We can therefore state out first Theorem and we can now also use the Law of Trichotomy in this way:

If we are working with a power of an Integer, only, the result of the Sum / Step Sum is independent from the K we choose:

If $A \in \mathbb{N}^+$ than we can write A^n as a TRIPLE EQUALITY:

$$A_N^n = \sum_{x=1}^{A} M_n = \sum_{x=1/K}^{A} M_{n,K} = \lim_{K \to \infty} \sum_{x=1/K}^{A} M_{n,K} = \int_0^A 3x^2 dx$$

Remembering that:

$$M_{n,K} = \binom{n}{1}\frac{x^{n-1}}{K} - \binom{n}{2}\frac{x^{n-2}}{K^2} + \binom{n}{3}\frac{x^{n-3}}{K^3} - ... + / - \frac{1}{K^n}$$

If $A_Q \in \mathbb{Q} - \mathbb{N} : A = P/K$ than we can write A_Q^n as:

$$A_Q{}^n < A_Q{}^n = \sum_{x=1/K}^{A_Q} M_{n,K} = \lim_{K \to \infty} \sum_{x=1/K}^{A_Q} M_{n,K} = \int_0^{A_Q} 3x^2 dx$$

If $A_R \in \mathbb{R} - \mathbb{Q}$ with $A_R =$ Irrational, than we can, in general, write A_R^n as:

$$A_N{}^n < A_Q{}^n = \sum_{x=1/K}^{A} M_{n,K} < A_R{}^n = \lim_{K \to \infty} \sum_{x=1/K}^{A_R} M_{n,K} = \int_0^{A_R} 3x^2 dx$$

And as told if the Irrationality of A_R depends on a known factor, for example $A_R = \sqrt{2} * A_Q$ where $A_Q = P/K \in (Q)$, than is possible to hack again the Step Sum step $s = (1/K) * \sqrt{2}$ to let it work with a finite number of Irrational Steps.
BUT, MORE IMPORTANT, we can BOUND A_Q AND A_R in this way:
If $A_Q \in \mathbb{Q} - \mathbb{N} : A_Q = P/K$ than we can write A_Q^n as:

$$A_N{}^n < (P/K)^n < (P+1)^n/K^n$$

If $A_R \in \mathbb{R} - \mathbb{Q}$ with $A_R = Irrational$, than we can, in general, pack A_R^n between the following Bounds, independently on how bug K is:

$$\sum_{x=1/K}^{A_Q} M_{n,K} < A_R{}^n < \sum_{x=1/K}^{A_Q+1/K} M_{n,K}$$

With the known conditions, THAT WE ALREADY KNOW THAT AT THE LIMIT FOR $K \to \infty$ JUST BOTH THE LIMIT CONVERGE TO ${A_R}^n$, so is again TRUE THE EQUALITY:

$$\lim_{K\to\infty} \sum_{x=1/K}^{A_Q} M_{n,K} = \int_0^{A_R} 3x^2 dx = \lim_{K\to\infty} \sum_{x=1/K}^{A_Q+1/K} M_{n,K}$$

This is the most interesting property we have seen till now, since this allow us to Re-Define using our Complicate Rational Modulus what an IRRATIONAL ROOT is. And we already have seen that this Convergence is NOT SMOOTH, but in media it is Monotone Convergent.

What is now, I hope more clear is that via Exact Calculus, so applying the rule for finding maximum and minimum, we can now easy find the abscissa of two following rationals having the same ordinate (in our polynomial function, for the moment).

But also, if we are smart enough to find the Proper Rational Derivation Step, to find one or all the Rational Roots of our Polynomial equation.

Chapt.9: How to work with Irrational values

There is a last interesting case: our Step Sum can be able to rise an irrational value $P \in \mathbb{R} - \mathbb{Q}$ just in case the Irrational Factors (be it a single one, or, more in general, an aggregation of Real Numbers we can qualify as an Irrational) can be taken out from the sum.
This is the key point I'll use in the Vol.2 to prove Fermat is right and he has in the hands all the "simple" instruments shown till here to state your Last theorem and prove it.
I know from several years of discussions on several different forums that while it's clear for everybody that (for example) if:

$$P = \pi * A^2$$

we can write is as:

$$P = \pi * A^2 = \pi * \sum_{1}^{A} 2x - 1$$

It's complicate for someone to understand that we can carry (for example) the Square Root of π into the Sum in this way:

$$P = \pi * A^2 = \pi * \sum_{1}^{A} 2x - 1 = \sum_{1*\sqrt{\pi}}^{A*\sqrt{\pi}} 2x - 1$$

Where we move of an Irrational step:

$$1 * \sqrt{\pi}, 2 * \sqrt{\pi}, ..., i * \sqrt{\pi}, ..., A * \sqrt{\pi}$$

Since as told and proved in the previous chapters, when we multiply the index for a certain value K means we multiply all the Sum by the n-th power of such K

As shown in the previous chapters this holds true also when the value K is a Rational $k = 1/K$

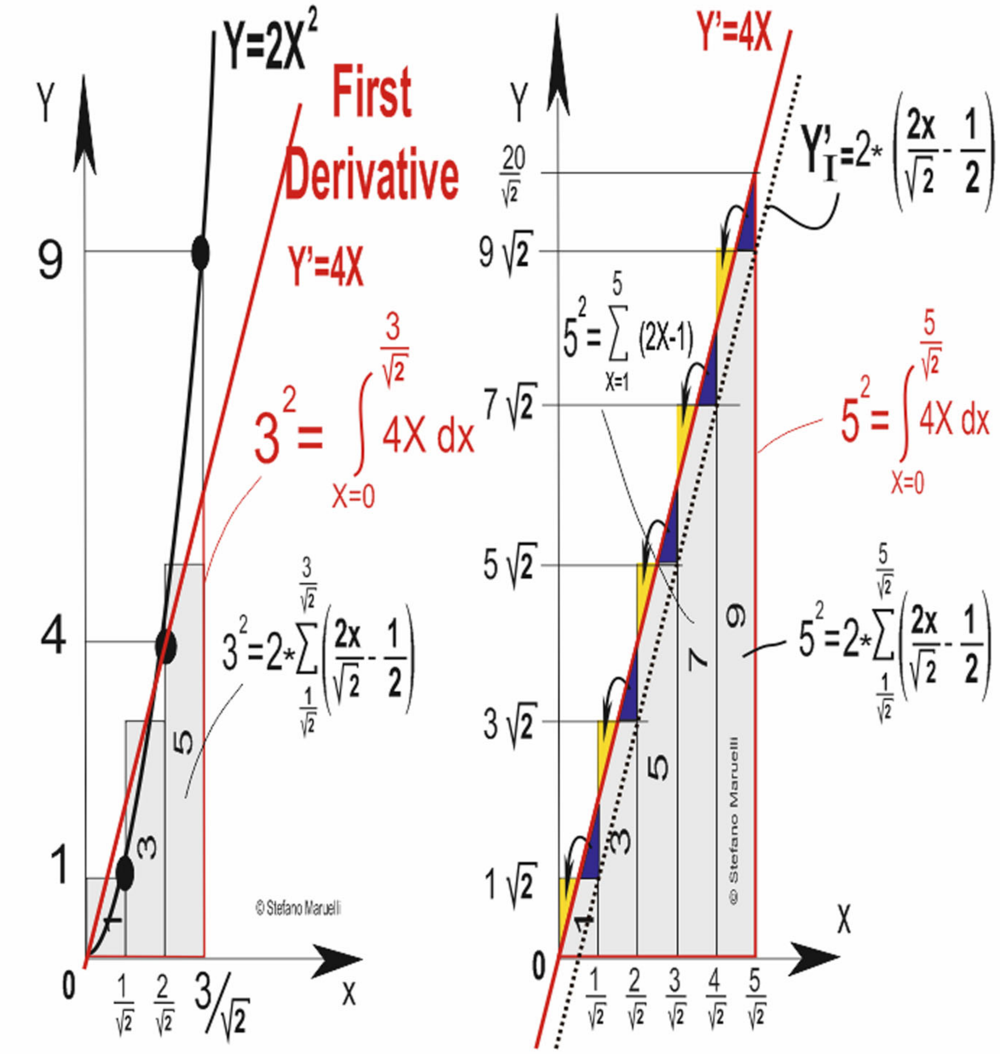
Here an example (and proof) of what happens in case we have $K = \sqrt{2}$. Taking $x = X/\sqrt{2}$:

$$P = a^2 = \frac{A^2}{2} = \frac{1}{2} \sum_{1}^{A} 2x - 1 = \sum_{x=1/\sqrt{2}}^{A/\sqrt{2}} \frac{2x}{\sqrt{2}} - \frac{1}{2} =$$

$$= 2 * \sum_{x=1}^{A} \left\{ \left(\frac{2x}{2^{(1/2)}} \cdot \frac{1}{2^{(1/2)}} \right) - \frac{1}{2} \right\}$$

Sqares Below Y=2X² First Derivative: Y'=4X

The ***First Derivative*** *is squared by* ***Irrational Gnomons***



X	x=X/sqrt(2)	Agm = Dx * Ygm =M2_k=sqrt(2):	Ygm = M2k*sqrt(2)	Sum = X^2
		Dx= x(i+1) - x(i)		
		2*M_2,K= 2*(2x/sqrt(2) -1/2)		
1	0.707106781	1	1.414213562	1
2	1.414213562	3	4.242640687	4
3	2.121320344	5	7.071067812	9
4	2.828427125	7	9.899494937	16
5	3.535533906	9	12.72792206	25

A is a positive integer, the sum has an integer number of summands, but we have a Step that is an irrational number.

We can write:

$$2*\sum_{x=1}^{a}\left\{\left(\frac{2x}{2^{(1/2)}}*\frac{1}{2^{(1/2)}}\right)-\frac{1}{2}\right\}=2*\sum_{x=1}^{a}\left(\frac{\cancel{2}x}{\cancel{2}}-\frac{1}{2}\right)=$$

$$2*\left(\sum_{x=1}^{a}x\right)-\sum_{x=1}^{a}1=\frac{\cancel{2}a(a+1)}{\cancel{2}}-a=a^2+a-a=a^2.$$

Another example in case n = 3:

$$2*\sum_{x=1}^{a}\left\{3*\left(\frac{x}{2^{(1/3)}}\right)^2*\frac{1}{2^{(1/3)}}-\left(\frac{3x}{2^{(1/3)}}*\frac{1}{2^{(2/3)}}\right)+\frac{1}{2}\right\}=3*\sum_{x=1}^{a}x^2-3*\sum_{x=1}^{a}x+\sum_{x=1}^{a}1=$$

$$=\frac{3a(a+1)(2a+1)}{6}-\frac{3a(a+1)}{2}+\frac{2a}{2}=\frac{2a^3+3a^2+a-3a^2-3a+2a}{2}=a^3.$$

This will becomes useful when in the Vol.2 I'll present my proof of Fermat the Last Theorem.

The point is always the same: If C is an Integer we can rise an Irrational Upper Limit Value, for example $C/2^{1/n}$ making an Integer Number of Step equal to C, at the condition that we use the right Irrational Step $K = 1/2^{1/n}$. So the base of the Gnomons has to perfectly divide the distance x from the origin, here $x = C/2^{1/n}$.

For the same reason is also true, for example that:

$$P=a^2=\frac{A^2}{2^{2/3}}=\sum_{x=1/2^{1/3}}^{A/2^{1/3}}\frac{2x}{2^{1/3}}-\frac{1}{2^{2/3}}$$

The same for any other Power following just the Power rules.

How to use different Irrational Complicate Modulus to represent the same value

I hope is clear now that we can use different Irrational Complicate Modulus to represent the same value.

For example we can use $M_{2,i}$ or any other $M_{n,i}$ to represent a Square.

Here a Square written via Cubic Irrational Modulus:

$$p^2 = \sum_{x=1/p^{1/3}}^{p} \frac{2x}{2^{1/3}} - \frac{1}{2^{2/3}} = \sum_{x=1/p^{1/3}}^{p/p^{1/3}} \frac{3x^2}{p^{1/3}} - \frac{3x}{p^{2/3}} + \frac{1}{p}$$

Unfortunately to dismount the Left hand using part of the Right hand terms (or vice versa) is possible but will not lead to the identity $0 = 0$ since we have different Triangles, but will lead to a solving equation we know has p as solution.
For those still can't believe to my Irrational Complicate Modulus Algebra here is the numerical example:

Table 6: Representing an Irrational Square (of 13), via Irrational Cubic Modulus

X	$x/13^{1/3}$	$(3*x^2/(13)^{1/3}) - 3*x/((13)^{2/3}) + 1/13$	Sum
1	0,42529037	0,076923077	0,0769231
2	0,850580741	0,538461538	0,6153846
3	1,275871111	1,461538462	2,0769231
4	1,701161481	2,846153846	4,9230769
5	2,126451851	4,692307692	9,6153846
6	2,551742222	**7**	16,615385
7	2,977032592	9,769230769	26,384615
8	3,402322962	**13**	39,384615
9	3,827613333	16,69230769	56,076923
10	4,252903703	20,84615385	76,923077
11	4,678194073	25,46153846	102,38462
12	5,103484443	30,53846154	132,92308
13	**5,528774814**	**36,07692308**	**169**

What is interesting is the (quite trivial) fact that some Term of the Sum is an integer value.

And it happens each time (using the same example) we have $\frac{X}{13^{1/3}}$ for what:

$$3*x^2/(13)^(1/3)) - 3*x/((13)^(2/3)) + 1/13 = integer$$

So when:

$$\left(\frac{3*X^2 - 3*X + 1}{13}\right) mod 13 = 0$$

in this case $X = 6$ gives:

$$\frac{3*6^2 - 3*6 + 1}{13} = 7$$

and $X = 8$ gives:

$$\frac{3*8^2-3*8+1}{13}=13$$

Again we have new series for Oeis.org, but unfortunately they no longer appreciate my so productive work...

But, most important, there is a know method to mathematically solve this congruence, remembering that:

$$ax^2+bx+c\equiv 0 \overset{\cdot 4a}{\iff} (2ax+b)^2\equiv b^2-4ac \pmod p$$

Chapt.10: Out from the Rational:

We can now try to revisit some classic math problem, for example how to prove that the n-th root of an Integer, is not a Perfect n-th power, is an Irrational. While the Classic Proof start from the fact that we must already know that the initial number is not a Perfect n-th power, so we need to make a numerical check of it before starting the proof, the proof via Complicate Modulus Algebra seems to me more direct.

Proof that $P^{(1/n)} =$ Irrational if $P \neq p^n$; $P, p \in \mathbb{N}$:

Standard Proof are based on the Initial Statement that $P \in \mathbb{N}$ or A is a Perfect Square. From some Algebra book you can find this sort of proof seems to me "more than obscure":

"Classic Proofs of Irrationality of $\sqrt{2}$:
A short proof of the Irrationality of $\sqrt{2}$ can be obtained from the Rational Root Theorem, that is, if $p(x)$ is a Monic polynomial with integer coefficients, then any Rational Root of $p(x)$ is necessarily an Integer.

Applying this to the polynomial $p(x) = x^2 - 2$, it follows that $\sqrt{2}$ is either an Integer or Irrational. Because $\sqrt{2}$ is not an integer (2 is not a perfect square), $\sqrt{2}$ must therefore be Irrational.

This proof can be generalized to show that any Root of any Natural Number which is not the Square of a Natural Number is Irrational.

I very disagree with this kind of 'proof' since it prove you nothing ..if you don't know what many other things are... And in case $A \in \mathbb{Q}$ and in case we don't know if P is a $PerfectSquare$, then we have no way to prove if $\sqrt{P}$ is an $Irrational$ or not."

But the above proof can looks circular (to one do not well understand it) since it seems it start assuming that an Irrational p is not the Root of a Perfect Power, then close saying it is for sure not an Integer since it Square is not the Square of an integer. So a student need to spend time on to understand how it works.

The Old proof looks more clear:
Suppose that $\sqrt{2}$ is a rational number. Then it could be written as

$$\sqrt{2} = \frac{p}{q}$$

for two natural numbers, p and q. Then squaring would give

$$2 = \frac{p^2}{q^2}$$

$$2q^2 = p^2$$

so 2 must divide ”p”². Because 2 is a prime number, it must also divide p, by Euclid's lemma. So $p = 2r$, for some integer r, But then

$$2q^2 = (2r)^2 = 4r^2$$

$$q^2 = 2r^2$$

which shows that 2 must divide q as well. So $q = 2s$ for some integer s. This gives

$$\frac{p}{q} = \frac{2r}{2s} = \frac{r}{s}$$

Therefore, if $\sqrt{2}$ could be written as a rational number, it could always be written as a rational number with smaller parts, which itself could be written with yet-smaller parts, "ad infinitum".

But for the "Well-ordering principle" this is impossible in the set of Natural Numbers. Since $\sqrt{2}$ is a Real Number, which can be either rational or irrational, the only option left is for $\sqrt{2}$ to be irrational.

(Alternatively, this proves that if $\sqrt{2}$ were Rational, no "smallest" representation as a fraction could exist, as any attempt to find a "smallest" representation p/q would imply a smaller one existed, which is a similar contradiction).

So here I try to present a proof using the Complicate Modulus Numbers.

Proof that if $P \in \mathbb{Q} - \mathbb{N}$ then $P^{1/n} \notin \mathbb{Q}$ so $P^{1/n}$ is an Irrational:

I can show with our new n-th Root method we can prove it without knowing in advance if P is a Perfect Power:

If $P^{1/n}\mathbb{Q} = Q/K$ and (both) $K = 10^m$ perfectly divide Q and Q/K has a Finite Number of Decimal Digits $m \in \mathbb{N}$, than we can transform P in an integer multiplying it by 10^m, than making our n-th root with the Recursive Difference modulo M_n we can have just two case:

The difference stops with a $Rest = 0$ or not.

In case we have $Rest = 0$ than $P^{1/n} \in \mathbb{Q}$, else if $Rest > 0$ than $P^{1/n} \notin \mathbb{Q}$.

Another Infinite Descent:

Another, unnecessary way, is to use the Rational Complicate Modulus $M_{n/K=10}$ to see that if $(P * 10^m)^{1/n} \in \mathbb{N}$ than the recursive difference stops exactly to $p = (P * 10^m)^{1/n}$, with $p \in \mathbb{N}$, so the first Decimal Digit (and all the following if we rise m) will be Zero. While if it doesn't, we will have one more significant Decimal Digit and at this point we are sure continuing to rise m we will have infinite many non periodic Decimal Digits.

But one can be unfamiliar with this kind of Modular Algebra and can ask for more details so we can complete the proof using the limit of the Sum for $K \to \infty$ so what is my Infinite Descent:
We know that:

$$\left(\sqrt{2}\right)^2 = 2 = \lim_{K\to\infty} \sum_{1/K}^{\sqrt{2}} \left(\frac{2x}{K} - \frac{1}{K^2}\right) = \int_0^{\sqrt{2}} 2x dx = 2$$

The very clear proof comes by the fact that we can define 2 Bound:

A **Lower Bound** having a Rational Upper Limit (Upper Limit for the Lower Bound) is $ULLB = Q/K < \sqrt{2}$ since we know is:

$$(\sqrt{2})^2 = 2 > \sum_{1/K}^{ULLB} \left(\frac{2x}{K} - \frac{1}{K^2}\right)$$

for any $K \in \mathbb{N}^+$

And an **Upper Bound**, since we rise a Rational Lower Limit for the Upper Bound (Lower Limit for the Upper Bound) $LLUB = ULLB + 1/K = Q/K + 1/K > \sqrt{2}$ since we know is:

$$(\sqrt{2})^2 = 2 < \sum_{1/K}^{ULLB+1/K} \left(\frac{2x}{K} - \frac{1}{K^2}\right)$$

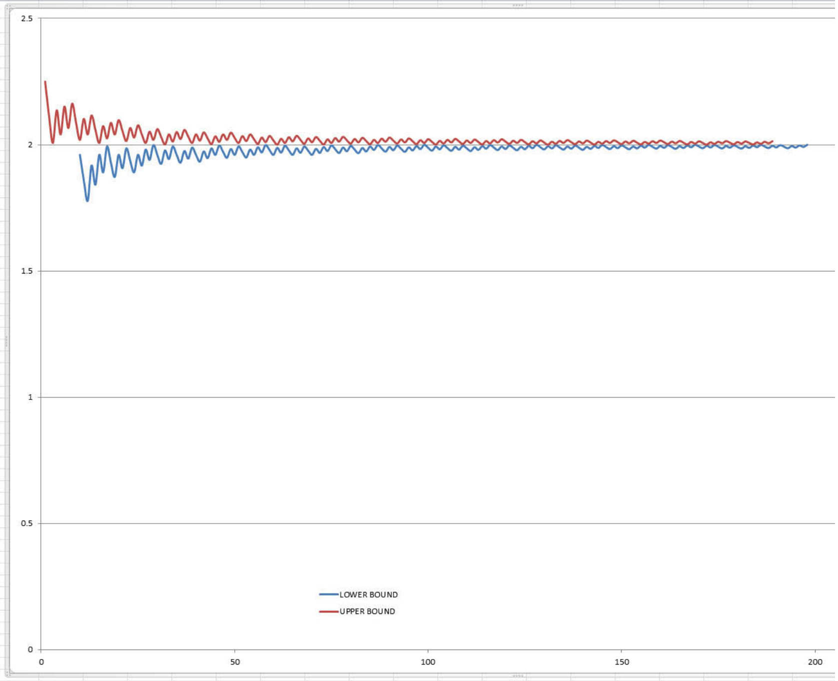
for any $K \in \mathbb{N}^+$

And we know by limits rules that both Upper and Lower Limits converge to $\sqrt{2}$ for $K \to \infty$

$$\sum_{1/K}^{ULLB} \left(\frac{2x}{K} - \frac{1}{K^2}\right) < (\sqrt{2})^2 < \sum_{1/K}^{ULLB+1/K} \left(\frac{2x}{K} - \frac{1}{K^2}\right)$$

Still if it isn't a smooth convergence, in fact if we plot the Best Rational Approximated Lower Value $ULL = Q/K < \sqrt{2}$ rising K from 1 to 10^m we will see a Saw Teeth function like the one below, due to the fact that some K divisor better approximate the Limit Value:

	RATIONAL APPROACH to \sqrt(2)		
K	x	ULL^2	(UUL)^2 = (ULL+1/K)^2
1	1	1	4
2	1	1	2.25
3	1.333333	1.77777778	2.777777778
4	1.25	1.5625	2.25
5	1.4	1.96	2.56
6	1.333333	1.77777778	2.25
7	1.285714	1.65306122	2.040816327
8	1.375	1.890625	2.25
9	1.333333	1.77777778	2.086419753
10	1.4	1.96	2.25
11	1.363636	1.85950413	2.115702479
12	1.333333	1.77777778	2.006944444
13	1.384615	1.91715976	2.136094675
14	1.357143	1.84183673	2.040816327
15	1.4	1.96	2.151111111
16	1.375	1.890625	2.06640625
17	1.411765	1.99307958	2.162629758
18	1.388889	1.92901235	2.086419753
19	1.368421	1.87257618	2.019390582
20	1.4	1.96	2.1025
21	1.380952	1.90702948	2.040816327
22	1.409091	1.98553719	2.115702479
23	1.391304	1.93572779	2.058601134
24	1.375	1.890625	2.006944444
25	1.4	1.96	2.0736
26	1.384615	1.91715976	2.025147929
27	1.407407	1.98079561	2.086419753
28	1.392857	1.94005102	2.040816327
29	1.413793	1.99881094	2.097502973
30	1.4	1.96	2.054444444
31	1.387097	1.92403746	2.014568158
32	1.40625	1.97753906	2.06640625
33	1.393939	1.94306703	2.028466483
34	1.411765	1.99307958	2.076989619
35	1.4	1.96	2.040816327
36	1.388889	1.92901235	2.006944444
37	1.405405	1.97516435	2.051862673
38	1.394737	1.94529086	2.019390582
39	1.410256	1.98882314	2.061801446
40	1.4	1.96	2.030625
41	1.390244	1.93277811	2.001189768
42	1.404762	1.97335601	2.040816327
43	1.395349	1.94699838	2.012439156
44	1.409091	1.98553719	2.050103306
45	1.4	1.96	2.022716049
46	1.413043	1.99669187	2.058601134
47	1.404255	1.971933	2.03214124
48	1.395833	1.94835069	2.006944444
49	1.408163	1.98292378	2.040816327
50	1.4	1.96	2.0164
51	1.411765	1.99307958	2.048827374
52	1.403846	1.97078402	2.025147929
53	1.396226	1.9494482	2.00249199
54	1.407407	1.98079561	2.033264746
55	1.4	1.96	2.011239669

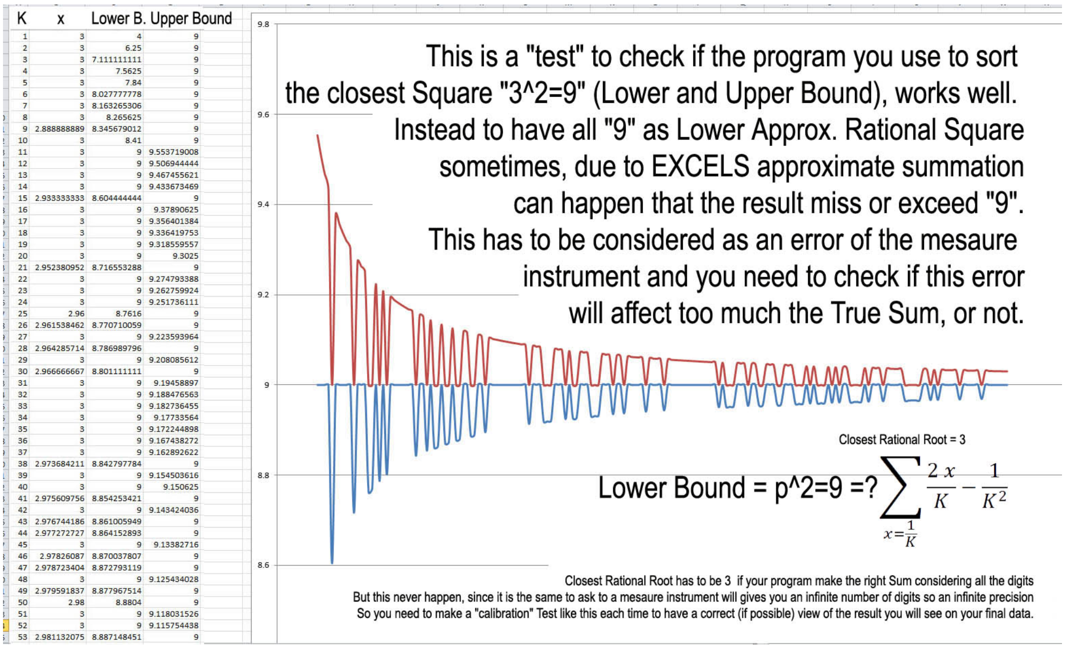


Of course you need to write some line of code to obtain the closest two Rational Values, and at the moment there is no LaTeX sing to define what is similar to the Floor / Ceeling operator, but that works with Rational, and as you can see will have infinite number of Values depending by the K you choose.

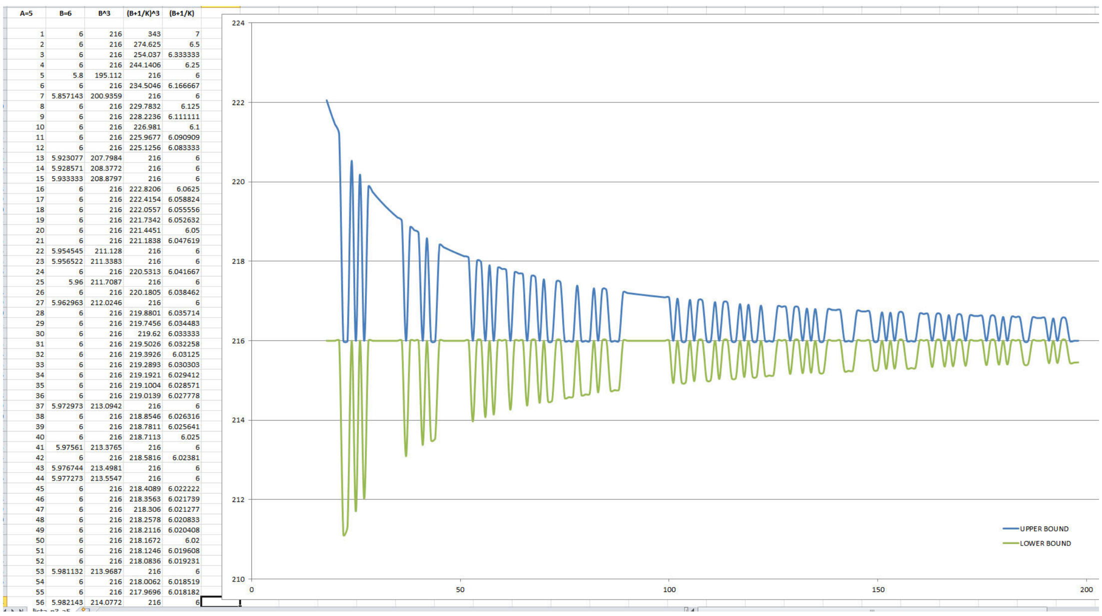
What is important here, after you wrote your program to sort the Value of the Sums rising K you need to build this graph

is to remember that any software works with a limited number of Digit, and this will affect your Measure by an error.

You can try to figure out how the error behave, making the same approach to a Known Genuine Square, here for example 9. As you can see the digit my VB Program take in count, will produce this error:
Looking at the value we see that are little enough to NOT disturb too much the real Measure, but for sure an error is present and we have to remember that the Sum of Two errors / approximations will not give an error that is exactly the Sum of the Two.



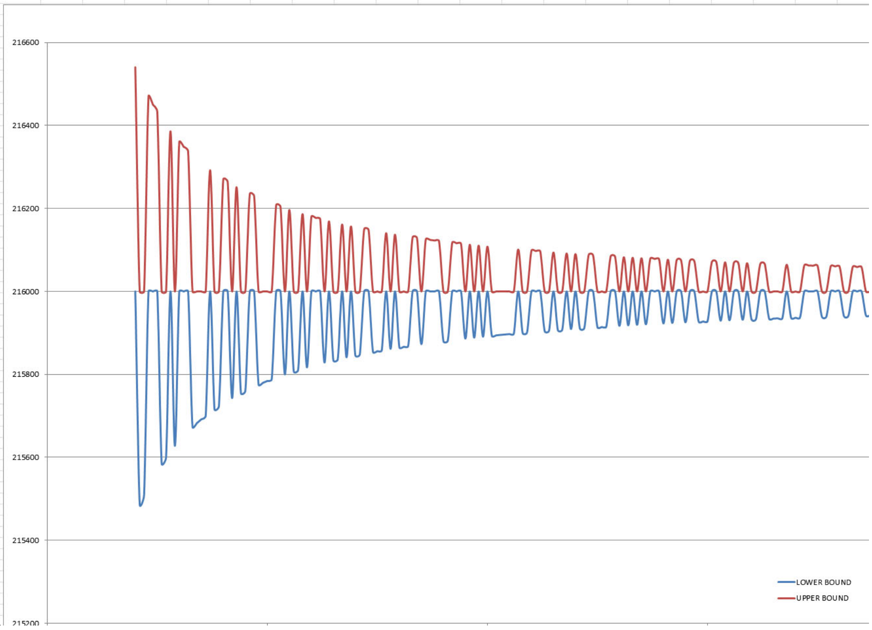
Here an example of the Error EXCEL did on the Step Sum, Step $1/K$ (K=1 to 1000) till $B^3 = 6^3$



Here an example of the Error EXCEL did on the Step Sum, Step $1/K$ (K=1 to 1000) till $B^3 = 60^3$. As you can see Measure Scale makes Lot of Difference !

All this will be very useful once we have to prove the FLT, but to prove the theorem we will need to find a way to generalize for a generic triplet value represented by Letters A,B,C, instead of a single known triplet of numbers, here for example $A = 5$, $B = 6$, for what for n=3 will happen that: $A^3 + B^3 = 341 \neq C^3 = 7^3 = 343$.

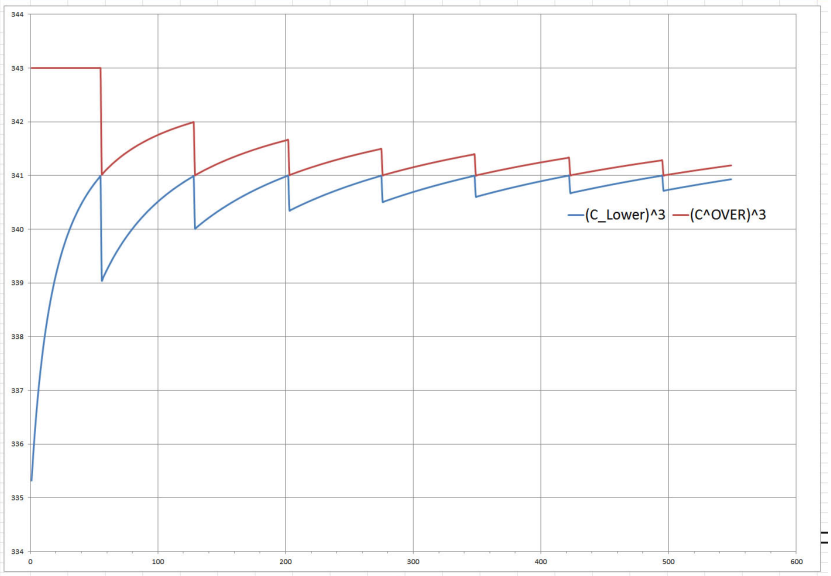
A=50	B=60	B^3=216000	(B+1/K)^3	(B+1/K)
1	60	216000	226981	61
2	60	216000	221445.1	60.5
3	59.66667	212419.963	216000	60
4	60	216000	218711.3	60.25
5	59.8	213847.192	216000	60
6	60	216000	217805	60.16667
7	59.85714	214460.8134	216000	60
8	60	216000	217352.8	60.125
9	59.88889	214802.2209	216000	60
10	59.9	214921.799	216000	60
11	59.90909	215019.6687	216000	60
12	59.91667	215101.2494	216000	60
13	59.92308	215170.2954	216000	60
14	60	216000	216772.3	60.07143
15	59.93333	215280.7997	216000	60
16	60	216000	216675.7	60.0625
17	59.94118	215365.3285	216000	60
18	59.94444	215400.5554	216000	60
19	59.94737	215432.0774	216000	60
20	60	216000	216540.5	60.05
21	59.95238	215486.1223	216000	60
22	59.95455	215509.4627	216000	60
23	60	216000	216469.9	60.04348
24	60	216000	216450.3	60.04167
25	60	216000	216432.3	60.04
26	59.96154	215584.8816	216000	60
27	59.96296	215600.2469	216000	60
28	60	216000	216385.9	60.03571
29	59.96552	215627.8002	216000	60
30	60	216000	216360.2	60.03333
31	60	216000	216348.6	60.03226
32	60	216000	216337.7	60.03125
33	59.9697	215672.8925	216000	60
34	59.97059	215682.5086	216000	60
35	59.97143	215691.5755	216000	60
36	59.97222	215700.1389	216000	60
37	60	216000	216292	60.02703
38	59.97368	215715.9141	216000	60
39	59.97436	215723.1952	216000	60
40	60	216000	216270.1	60.025
41	60	216000	216263.5	60.02439
42	59.97619	215742.9592	216000	60
43	60	216000	216251.3	60.02326
44	59.97727	215754.6384	216000	60
45	59.97778	215760.0889	216000	60
46	60	216000	216234.9	60.02174
47	60	216000	216229.9	60.02128
48	59.97917	215775.0781	216000	60
49	59.97959	215779.6668	216000	60
50	59.98	215784.072	216000	60
51	59.98039	215788.3045	216000	60
52	60	216000	216207.8	60.01923
53	60	216000	216203.8	60.01887
54	59.98148	215800.0617	216000	60
55	60	216000	216196.4	60.01818
56	59.98214	215807.2002	216000	60



A=5 B=6 Approach example to (A^3+B^3)=341 =?C^3

C^3 Lower=\sum_{1/K}^{C_Lower} 3x^2/K-3x/K^2+1/K^3

K	C_Lower	C^Over	(C_Lower)^3	(C^OVER)^3
1	6	7	216	343
2	6.5	7	274.625	343
3	6.666666667	7	296.2962963	343
4	6.75	7	307.546875	343
5	6.8	7	314.432	343
6	6.833333333	7	319.0787037	343
7	6.857142857	7	322.425656	343
8	6.875	7	324.9511719	343
9	6.888888889	7	326.9245542	343
10	6.9	7	328.509	343
11	6.909090909	7	329.809166	343
12	6.916666667	7	330.8952546	343
13	6.923076923	7	331.8161129	343
14	6.928571429	7	332.6067784	343
15	6.933333333	7	333.293037	343
16	6.9375	7	333.8942871	343
17	6.941176471	7	334.425402	343
18	6.944444444	7	334.8979767	343
19	6.947368421	7	335.3211838	343
20	6.95	7	335.702375	343
21	6.952380952	7	336.0475111	343
22	6.954545455	7	336.3614763	343
23	6.956521739	7	336.648311	343
24	6.958333333	7	336.911386	343
25	6.96	7	337.153536	343
26	6.961538462	7	337.377162	343
27	6.962962963	7	337.5843113	343
28	6.964285714	7	337.7767402	343
29	6.965517241	7	337.9559638	343
30	6.966666667	7	338.1232963	343
31	6.967741935	7	338.2798832	343
32	6.96875	7	338.4267273	343
33	6.96969697	7	338.5647105	343
34	6.970588235	7	338.6946112	343
35	6.971428571	7	338.8171195	343
36	6.972222222	7	338.9328489	343
37	6.972972973	7	339.0423469	343
38	6.973684211	7	339.1461037	343
39	6.974358974	7	339.2445591	343
40	6.975	7	339.3381094	343
41	6.975609756	7	339.4271122	343
42	6.976190476	7	339.5118913	343
43	6.976744186	7	339.5927403	343
44	6.977272727	7	339.6699263	343
45	6.977777778	7	339.7436927	343
46	6.97826087	7	339.8142619	343
47	6.978723404	7	339.8818374	343
48	6.979166667	7	339.9466055	343
49	6.979591837	7	340.0087379	343
50	6.98	7	340.068392	343
51	6.980392157	7	340.1257133	343
52	6.980769231	7	340.1808361	343
53	6.981132075	7	340.2338843	343



A turn in the real life of measuring:

What is also interesting, for those is not familiar with Measure theory, and art, is that in case we are making a real Measure with an instrument that has a precision K, and one wanna obtain the Best Possible Result (having a finite budget) is NOT at all sure that spending more for a more "precise" instrument, so with a Bigger K, one have back , for sure, a Better Measure or the Better one.

In fact in this last example one having an instrument with precision $K = 73$

measuring the the Irrational Root coming from the value of 341

with an instrument capable to Measure just Rational Powers n=3 with the $M_{n,K}$ modulus,

will obtain 340.9902395,

while one having an instrument with precision $K = 954$

have back a worst 340.8472866.

And since the cost of a 12 times more precise instruments is several times more than 12 times, just, (and sometimes it is more than 12^2 or out of any budget at all if physically impossible to be built with the actual tech) , one can immediately understood how is important to choose the right K while making a Measure, so to choose the right instrument. For those are familiar with electronic device you know the problem of measuring the Voltage on a circuit, over a resistance: depending if we have an high or big resistance we need to evaluate if use a voltmeter (that has an high internal resistance) or an ammeter, that has a low resistance.

Understood what above, will be easy to write another Proof:

Proof of Legendre Conjecture:

I hope what follows will prove the Legendre's conjecture:
Be: π_i the i-esim Prime Number, then

$$\sum_{1/K}^{q}\left(\frac{2x}{K}-\frac{1}{K^2}\right) < (\sqrt{\pi_i}) < \sum_{1/K}^{q+1/K}\left(\frac{2x}{K}-\frac{1}{K^2}\right)$$

Where I hope you already are, now, familiar with my Step Sum notation.

2) A clarification of what let the inequality be true, comes from the easy concerning that for any $R \in \mathbb{R} - \mathbb{Q}$ is true that:

$$\int_0^R M_{n,K,x}\ dx < \int_0^R nX^n dx$$

and

$$\int_0^R M_{n,K,x+1}\ dx > \int_0^R nX^n dx$$

for any $K \in \mathbb{Q}^+$

- So the proof comes showing that since the Limit exists and is our Irrational (square),

and that till K is not at his Limit (∞), Both the Terms (all the terms), still if calculate as an Integral between 0 and R, or as a Sum between $1/K$ and R, show true the inequality because they are

LITTLEST THEN THE ONES AT THEIR LIMIT FOR $K \to \infty$ ARE.

In short till we have a Gear (instead of a smooth circle) we cannot rise an Irrational Upper Limit or a Limit that is an aggregation of Irrationals (will not produce a much or less trivial cancellation).

3) But the theorem can be extended to all the Irrationals R and to all the aggregation of Irrationals (sum, product, etc...) WE CAN PROVE will not produce trivial cancellation(s) and are for so Proven Irrationals.

Trivial cancellation are for example:

$p + q = (1 + \sqrt{2}) + (1 - \sqrt{2})$

$p * q = (\sqrt{2}) * (1/\sqrt{2})$

WHILE WE HAVE TO TAKE LOT OF CARE WITH NON TRIVIAL (Known) CANCELLATION, that are probably Wrong.

So before prove that $e + \pi$ or 2^e and several other tricks with Irrationals, are for sure Irrationals too, we need to reflect about the fact that Limits will not always returns the expected simple/clear result, so we can divide them in two Classes:

1- The one having Convergent Bounds (also non smooth but in some ways strictly convergent):

so calling LB the $LowerBound$ and UB the $UpperBound$

for any x, is

$y_{LB} < Limit$ and $y_{UB} > Limit$,

(or in the known notation $y_L B = Limit - \epsilon$ and $y_U B = Limit + \epsilon$ for any arbitrary little $\epsilon > 0$...)

2- And the one having NO Convergent Bounds:
For example be:

$$\sum_{1}^{UL=\infty} a_i$$

with

$a_i = 1$ for $i = odd$

and

$a_i = -1$ for $i = even$

it is known it has an Indeterminate Limit, since IT IS NOT POSSIBLE to establish how such sequence will "end" at ∞: in fact for $UL \in N^{+}$ the answer is 1, or 0 depending if the $UL = Odd$ or $UL = Even$, but the reason can be better argued now:

Since there are No vanishing terms in this construction, is Not Possible to Define Converging Bounds, so WE CANNOT ASSIGN TO THIS SERIES A GENUINE VALUE AT ITS LIMIT (as already well known)
What is now, I hope more clear is that via Exact Calculus, so applying the rule for finding maximum and minimum, we can now easy find the abscissa of two following Irrationals (too) having the same ordinate (in our polynomial function, for the moment).

But also, if we a smart enough to find the Proper Irrational Derivation Step, to find one or all the Irrational Roots, too, of our Polynomial equation.

Complicate Modulus Algebra on the Imaginary Plane:

Root of a Negative Number:

Taking $\sqrt{-1}$ as example we can tray to extract this square Root with my Recursive Difference for Genuine Squares Modulus, so using the $M_{2,K,-} = 2x/K - 1/K$ we get a wrong result:

X	x=X/10	2x/10-1/10^2	SUM	Diff.	
			Square=	-1	
-1	-0.1	-0.03	-0.03	-0.97	
-2	-0.2	-0.05	-0.08	-0.89	
-3	-0.3	-0.07	-0.15	-0.74	
-4	-0.4	-0.09	-0.24	-0.5	
-5	**-0.5**	-0.11	-0.35	**-0.15**	**TOO LITTLE**
-6	**-0.6**	-0.13	-0.48	**0.33**	**TOO BIG**

X	x=X/10	2x/10^2-1/10^4	SUM	Diff.	
			Square=	-1	
-1	-0.01	-0.0003	-0.0003	-0.9997	
-2	-0.02	-0.0005	-0.0008	-0.9989	
-3	-0.03	-0.0007	-0.0015	-0.9974	
-4	-0.04	-0.0009	-0.0024	-0.995	
-5	-0.05	-0.0011	-0.0035	-0.9915	
-6	-0.06	-0.0013	-0.0048	-0.9867	
...	...	...	...	...	
-25	-0.25	-0.0051	-0.0675	-0.3825	
-26	-0.26	-0.0053	-0.0728	-0.3097	
-27	-0.27	-0.0055	-0.0783	-0.2314	
-28	-0.28	-0.0057	-0.084	-0.1474	
-29	**-0.29**	-0.0059	-0.0899	**-0.0575**	**TOO LITTLE**
-30	**-0.3**	-0.0061	-0.096	**0.0385**	**TOO BIG**

X	x=X/10^3	2x/10^3-1/10^6	SUM	Diff.	
			Square=	-1	
-1	-0.001	-0.000003	-0.000003	-0.999997	
-2	-0.002	-0.000005	-0.000008	-0.999989	
-3	-0.003	-0.000007	-0.000015	-0.999974	
-4	-0.004	-0.000009	-0.000024	-0.99995	
-5	-0.005	-0.000011	-0.000035	-0.999915	
-6	-0.006	-0.000013	-0.000048	-0.999867	
...	...	...	...	...	
-133	-0.133	-0.000267	-0.017955	-0.189099	
-134	-0.134	-0.000269	-0.018224	-0.170875	
-135	-0.135	-0.000271	-0.018495	-0.15238	
-136	-0.136	-0.000273	-0.018768	-0.133612	
-137	-0.137	-0.000275	-0.019043	-0.114569	
-138	-0.138	-0.000277	-0.01932	-0.095249	
-139	-0.139	-0.000279	-0.019599	-0.07565	
-140	-0.14	-0.000281	-0.01988	-0.05577	
-141	-0.141	-0.000283	-0.020163	-0.035607	
-142	**-0.142**	-0.000285	-0.020448	**-0.015159**	**TOO LITTLE**
-143	**-0.143**	-0.000287	-0.020735	**0.005576**	**TOO BIG**

But this is just because, we know, the Root of -1 lies on another plane that is not the X-Y one, so it is not a Real Number. But the power of my Complicate Modulus Algebra is that there is an appropriate modulus perfectly extract the Square Root of -1 (and any n-th root of negative numbers) using the proper Imaginary Complicate (Rational) Modulus $M_{n,K,i}$.

To have the good result we have to change plane so keep the right Versor j (since we are no longer talking of Real Ordinate Y, but imaginary one we usually call i), and this happen simply changing all the sign of the Terms of the Complicate Modulus:

$$\sum_{j=1/K}^{1}\left(\frac{-2j}{K}+\frac{1}{K^2}\right)=-1$$

A	B	C	D	E	F	G	H	I
j	j/K K=10	-j/K+1/K^2	SUM		j	j/K K=100	-j/K+1/K^2	SUM
1	0.1	-0.01	-0.01		1	0.01	-0.0001	-0.0001
2	0.2	-0.03	-0.04		2	0.02	-0.0003	-0.0004
3	0.3	-0.05	-0.09		3	0.03	-0.0005	-0.0009
4	0.4	-0.07	-0.16		4	0.04	-0.0007	-0.0016
5	0.5	-0.09	-0.25		5	0.05	-0.0009	-0.0025
6	0.6	-0.11	-0.36		6	0.06	-0.0011	-0.0036
7	0.7	-0.13	-0.49		7	0.07	-0.0013	-0.0049
8	0.8	-0.15	-0.64		8	0.08	-0.0015	-0.0064
9	0.9	-0.17	-0.81		9	0.09	-0.0017	-0.0081
10	1	-0.19	-1		10	0.1	-0.0019	-0.01
11	1.1	-0.21	-1.21		11	0.11	-0.0021	-0.0121
12	1.2	-0.23	-1.44		12	0.12	-0.0023	-0.0144
					..	..	..	..
					94	0.94	-0.0187	-0.8836
					95	0.95	-0.0189	-0.9025
					96	0.96	-0.0191	-0.9216
					97	0.97	-0.0193	-0.9409
					98	0.98	-0.0195	-0.9604
					99	0.99	-0.0197	-0.9801
					100	1	-0.0199	-1
					101	1.01	-0.0201	-1.0201
					102	1.02	-0.0203	-1.0404
(c) Stefano Maruelli					103	1.03	-0.0205	-1.0609

And in this case is again True the Equality at the Limit, in fact:

$$\lim_{K\to\infty}\sum_{j=1/K}^{1}\frac{-2j}{K}+\frac{1}{K^2}=\int_{j=0}^{1}-2j\,dj=-2/2[j^2]_0^1=-1$$

So we can deal now with a Real Complicate Number, and with a Imaginary Complicate Number, that as the known one can be made by two part: a Real one plus a Complex one, and funny story, we again have our Talking Index that is now a Versor, x for Real, j for imaginary remembering one must be 90 degree respect to the other since we can see hereafter that if we try to force a negative Index into the Sum, we no longer have back an n-th Power.

In this way holds true the same Rule for the Integer/Rational/infinitesimal (here for square) we already seen in the previous chapters.:

$$P=\sum_{J=1}^{|p|}-2j+1=\sum_{j=1/K}^{|p|}\frac{-2j}{K}+\frac{1}{K^2}=\lim_{K\to\infty}\sum_{j=1/K}^{|p|}\frac{-2j}{K}+\frac{1}{K^2}=\int_{j=0}^{|p|}-2j\,dj$$

So, most in general, the Imaginary Complicate Modulus is:

$$M_{n,J}=-(J^n-(J-1)^n)$$

and follows that to have the Rational one we will call: $M_{n,J,K}$ is enough to change the sign of all the terms of the known Rational one $M_{n,K}$

Now the interesting question (for me):

- is "J" the imaginary Versor on witch Imaginary Roots of Negative Numbers Lies On ?

- or we can use X instead of J since it is the "continuation" of the same known X Versor ?

Will be enough to write a little collection of numerical example to have back the answer: we have just to try to keep J Positive or Negative, and to make all the possible exchange of Signs into the Modulus formula:

Table 7: TRYING TO USE A NEGATIVE INDEX and THE SAME MODULUS $M_{2,J} = -2J + 1$

J	**-2J+1**	**SUM**	**DIFF. -119**
-1	3	3	-122
-2	5	8	-127
-3	7	15	-134
-4	9	24	-143
-5	11	35	-154
-6	13	48	-167
-7	15	63	-182

So this is not correct. As will be wrong to use the (classic) COMPLICATE INTEGER MODULUS $M_2 = 2X - 1$:

Table 8: TRYING TO USE A NEGATIVE INDEX and THE (classic) COMPLICATE INTEGER MODULUS $M_2 = 2X - 1$

X	**2X-1**	**SUM**	**DIFF. 119**	
-1	-3	-3	122	
-2	-5	-8	127	
-3	-7	-15	134	
-4	-9	-24	143	
-5	-11	-35	154	????

While for $n = 2$ we can use the IMAGINARY SPECULAR MODULUS $M_{2,+} = 2X + 1$ and it Works:

Table 9: USING A NEGATIVE INDEX and IMAGINARY SPECULAR MODULUS $M_{2,+} = 2X + 1$

X	**2X+1**	**SUM**	**DIFF. -119**		
-1	-1	-1	-118		
-2	-3	-4	-115		
-3	-5	-9	-110		
-4	-7	-16	-103		
-5	-9	-25	-94		
-6	-11	-36	-83		
-7	-13	-49	-70		
-8	-15	-64	-55		
-9	-17	-81	-38		
-10	-19	-100	-19	<- P=-10i	Rest -19
-11	-21	-121	2		

While if we go higher for ODD powers we can see we have a wrong result. This is the classic Cubic Root:

Table 10: Add caption

J	**-3J2̂+3J-1**	**SUM**	**DIFF. -139**	**CUBIC ROOT**
1	-1	-1	-138	
2	-7	-8	-131	
3	-19	-27	-112	
4	-37	-64	-75	
5	-61	-125	-14	<- P=-5i
6	-91	-216	77	

While this is the Wrong one using the Negative Index and the Imaginary Complicate Modulus:

Table 11: USING A NEGATIVE INDEX and the IMAGINARY MODULUS $M_{3,+} = -3 * J^2 + 3 * J - 1$

J	**-3J2̂+3J-1**	**SUM**	**DIFF. -139**	**CUBIC ROOT**
-1	-7	-7	-132	
-2	-19	-19	-120	
-3	-37	-37	-102	
-4	-61	-61	-78	
-5	-91	-91	-48	
-6	-127	-127	-12	???
-7	-169	-169	30	

So in this way we still arrive to a Root, but is not the right one.

Chapt.11: Relations between M_n and $M_{n+/-1}$:

A nice property of the Complicate modulus M_n is that:

$$M_{n+1} = (n+1)\int(M_n) + C$$

Where C is the Integration Constant

For those are familiar with *Ordinals* it's clear I discovered a 2-th level of Order in Powers. We cannot just sort $\mathbb{N}$ and a bijection with our Gnomons M_n for one n we choose, but ALL of them, regardless of which n, are elements of a well sorted set $\mathbb{M}$, and the relation is bidirectional so it's also true for the derivative. So we can call this a Multidimensional Ordinal.
For example:

$$M_2 = 2x - 1$$

then:

$$M_3 = (n+1)\int(M_n) + C = 3\int(2x-1) = 3 * (2/2x^2 - x) + C$$

C will be $+1$ in case n is ODD,

C will be -1 in case n is EVEN

So:

$$M_3 = 3x^2 - 3x + 1$$

And so on.

The Proof is easy and follow the well known integration rules.

Another property of the Gnomons M_n is that:

$$M_{n-1} = \frac{1}{n} * \frac{d}{dx}(M_n)$$

So

$$M_2 = \frac{1}{n} * \frac{d}{dx}(M_n) = \frac{1}{3} * \frac{d}{dx}(3x^2 - 3x + 1) = 2x - 1$$

Chapt.12: How to "LINEARIZE" the n-th Problems

Continuing to show the rules to manipulate the sum without changing the result, I'll show here how to apply the previous RULES, in a SUM manipulation that allows us to Write Any POWER of Integers $Y = X^n$ $n >= 3$ as a SUM of LINEAR TERMS. So we can easily transform any n-th problem that involves just pure powers, in a linear problem.

I call this method "Linearization" for the reason it involves just linear terms, but also for other reasons that will be immediately clear once some more rules will be presented.

Rule7: Any N-th power of integers (from $n \geq 3$) is equal to a Sum of Linear Terms:

If n is EVEN ($n = 2p$; $n \geq 3$):

$$A^n = \sum_{x=1}^{A^{n/2}} (2x - 1) \tag{1}$$

If n is ODD ($n = 2p + 1$; $n \geq 3$) :

$$A^n = \sum_{x=1}^{A^{(n-1)/2}} (2xA - A) \tag{2}$$

1) Proof of the Rule7 in case n is EVEN (n=2p)

The only conditions for the (1) is $A \in \mathbb{N}^+$, so we have:

$A^n = A^{2p} = (A^p)^2 = B^2$ but we know that:

$$B^2 = \sum_{x=1}^{B} (2x - 1) = \sum_{x=1}^{A^p} (2x - 1) = \sum_{x=1}^{A^{n/2}} (2x - 1) = A^n$$

2) Proof of the Rule7 in case "n" is ODD (n=2p+1) So in this case is possible to re-arrange the sum to have:

$$A^n = A^{2p+1} = A^{2p} * A$$

For what we see just above this is equal to:

$$\sum_{x=1}^{A^p} (2x - 1) * A = \sum_{x=1}^{A^{(n-1)/2}} (2xA - A) = A^n$$

So it means that with this "LINEARIZATION" we can transform any Powers Problem in a Linear System problem. I'll investigate in the Beal conjecture inthe Vol.2, since other properties has to be shown.

Linearization is equal to an exchange of variable

Linearization is equal to an exchange of variable, for what is true the following General Rule, in case we would like to arrive to a Square:

In case of an ODD Power $Y = X^{2m+1}$ we can write:

$$Y = X^{2m+1} = X * X^{2m} = X * x^2$$

so the exchange we operate is:

$$x = \sqrt{\left(\frac{X^{2m+1}}{X}\right)} = X^m$$

In case of an Even Power:

$$Y = X^{2m} = X * x^2$$

so the exchange we operate is:

$$x = \sqrt{\frac{X^{2m}}{X}} = \frac{X^m}{\sqrt{X}}$$

To better understand the consequences of this Exchange of variable we need to investigate more in the next chapters in the field of certain Irrationals.

How to rewrite a Linearized Odd Power, taking the Base Factor, into the Index

Following the previous rules is possible to re-arrange the Sum an Odd Power taking the Base Factor, into the Index, to have:

$$A^n = \sum_{X=1}^{A^{(n-1)/2}} (2XA - A) = \sum_{x=1/A}^{A^{(n-1)/2}} \left(2x - \frac{1}{A}\right)$$

So for example for $A = 5$ we can write $A^3 = 125$ as:

Table 12: A^3 Linearized, Including the Base Factor into the Rational Index

$A = 5$		$A^3 = 125$			$A^3 = 125$
X	**2AX-A**	**SUM**	**x=X/A**	**2X-1/A**	**SUM**
1	5	5	0.2	0.2	0.2
2	15	20	0.4	0.6	0.8
3	25	45	0.6	1	1.8
4	35	80	0.8	1.4	3.2
5	45	125	1	1.8	5
6			1.2	2.2	7.2
7			1.4	2.6	9.8
8			1.6	3	12.8
9			1.8	3.4	16.2
10			2	3.8	20
11			2.2	4.2	24.2
12			2.4	4.6	28.8
13			2.6	5	33.8
14			2.8	5.4	39.2
15			3	5.8	45
16			3.2	6.2	51.2
17			3.4	6.6	57.8
18			3.6	7	64.8
19			3.8	7.4	72.2
20			4	7.8	80
21			4.2	8.2	88.2
22			4.4	8.6	96.8
23			4.6	9	105.8
24			4.8	9.4	115.2
25			5	9.8	125

And from now we can have a big suspect that Fermat is right since rewriting his famous Equation in sum we now have:

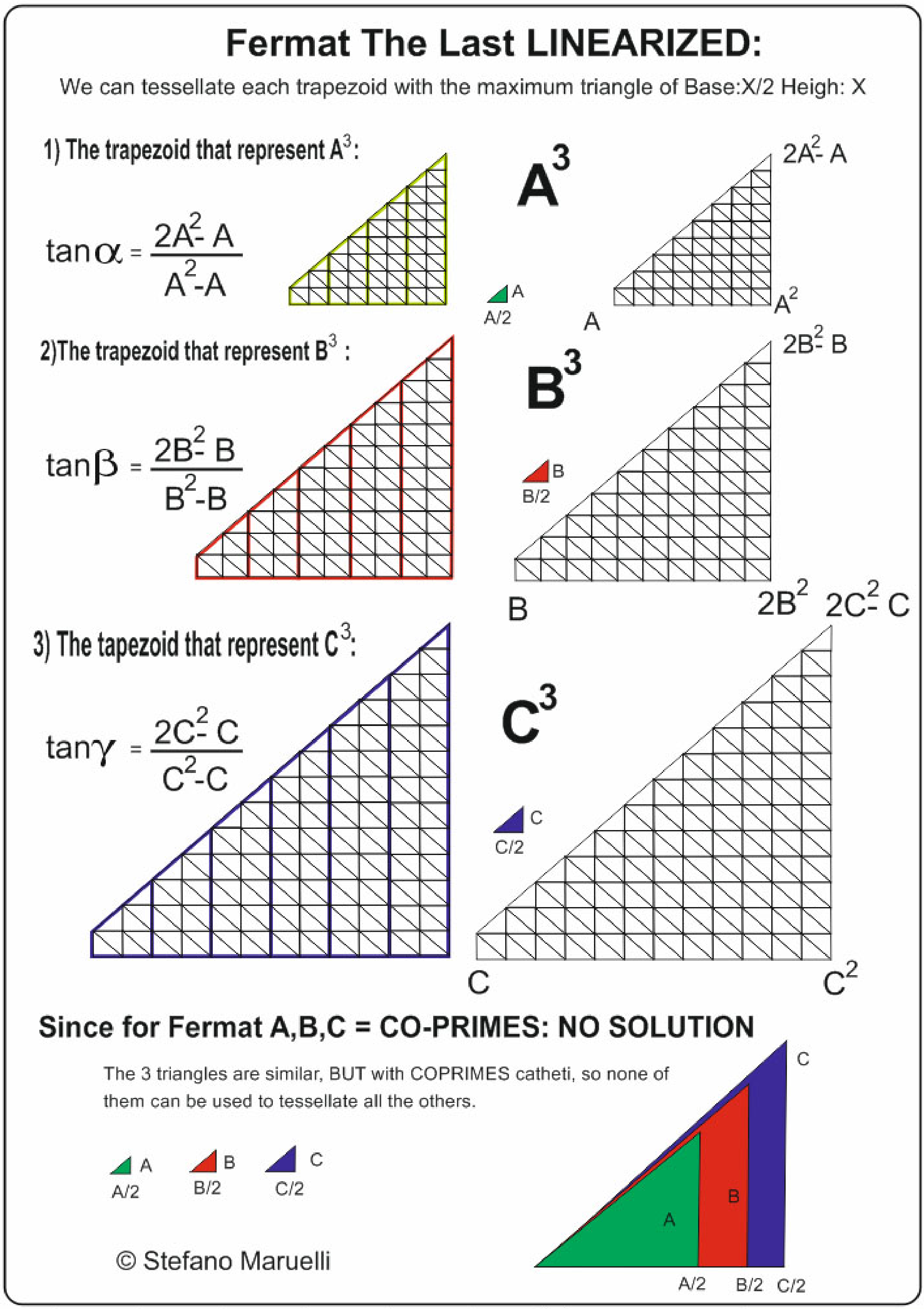
$$A^3 + B^3 =?C^3$$

$$\sum_{x=1/A}^{A^{(n-1)/2}} (2x - 1/A) + \sum_{x=1/B}^{B^{(n-1)/2}} (2x - 1/B) =? \sum_{x=1/C}^{C^{(n-1)/2}} (2x - 1/C)$$

Where we will prove in the Vol.2, there is no way to re-arrange any of the free parameter to let the Equality holds true in the Rational.

As usual we can see what we are trying to do on the Cartesian Plane once we paint the 3 Powers as 3 Trapezes:

Still going Rational with the tessellation there is no way to have the equality (the genuine proof will be given into the Vol.2)


Fermat The Last LINEARIZED:
We can tessellate each trapezoid with the maximum triangle of Base:X/2 Heigh: X
1) The trapezoid that represent A3:
tanα = (2A2- A)/(A2-A)
A3
2A2- A
A
A/2
A
A2
2)The trapezoid that represent B3 :
tanβ = (2B2- B)/(B2-B)
B3
2B2- B
B
B/2
B
2B2
2C2- C
3) The tapezoid that represent C3:
tanγ = (2C2- C)/(C2-C)
C3
C
C/2
C
C2
Since for Fermat A,B,C = CO-PRIMES: NO SOLUTION
The 3 triangles are similar, BUT with COPRIMES catheti, so none of
them can be used to tessellate all the others.
A
A/2
B
B/2
C
C/2
C
B
A
A/2
B/2
C/2
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Chapt.13: From x^n to x! via Recursive Difference

WI show now how x^n is connected to $n!$ due to what (I just discover few months ago) are known as Nexus Numbers. But I discover also that there are new hidden numbers, you can call Maruelli's Numbers or The Ghost Nexus Numbers.

	A	B	C	D	E	F	G
1							
2	**n=2**	**X**	**Y= X^2**	**Delta1= C^n-(C-1)^n**	**Delta2**	**Delta3**	
3		1	1				
4		2	4	3			
5		3	9	5	**5-2= 2**		
6		4	16	7	**7-5= 2**	0	
7		5	25	9	**2**	0	
8		6	36	11	**2**	0	
9		7	49	13	**2**	0	
10		8	64	15	**2**	0	
11		9	81	17	**2**	0	
12		10	100	19	**2**	0	
13		11	121	21	**2**	0	
14		12	144	23	**2**	0	
15		13	169	25	**2**	0	
16		14	196	27	**2**	0	
17				**n=2 so**	**n! = 2*1= 2**		
18	this is the only case where Delta1=Delta(n-1)						
19	and this is the reason why Fermat works						
20							
21	**n=3**	**X**	**Y= X^3**	**Delta1= C^n-(C-1)^n**	**Delta2**	**Delta3**	**Delta4**
22		1	1	Delta = 3x^2+3x+1	D2= 6x	D3= 6	
23		2	8	7			
24		3	27	19	12		
25		4	64	37	18	**18-12= 6**	
26		5	125	61	24	**6**	0
27		6	216	91	30	**6**	0
28		7	343	127	36	**6**	0
29		8	512	169	42	**6**	0
30		9	729	217	48	**6**	0
31		10	1000	271	54	**6**	0
32		11	1331	331	60	**6**	0
33		12	1728	397	66	**6**	0
34		13	2197	469	72	**6**	0
35		14	2744	547	78	**6**	0
36					**n=3 so**	**n! = 3*2*1 = 6**	
37	From this case Delta1 <> Delta(n-1)						
38	and this is the reason why Fermat doesen't works						
39	But we can see that the Delta(n-1) difference permit us to make an association between Xn and n!:						
40							

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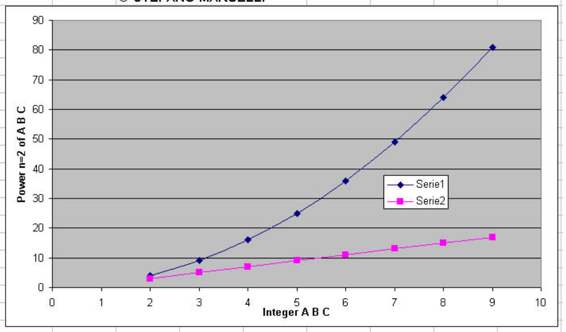
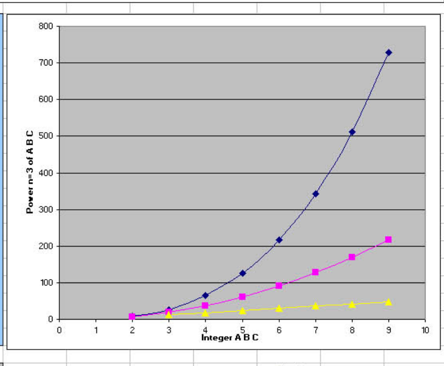




Foglio1 Foglio2 Foglio3

Recursive differences take us from x^n to $n!$

I hope the above table is readable raw by raw, to see that if we call δ^* the difference between two raws of the same column:

$\delta 1 = C_x - C_{x-1}$,(remembering that so far we call $M_n = \delta 1 = (x^n - (x-1)^n)$,

going ahead on the same raw, so making the next new columns with the new difference between two following values of the previous column:

$\delta 2 = D_x - D_{x-1}$; and th same for $\delta 3 ... \delta(n-1)$

We make the same process we did making the following derivative of a function, till we arrive to a fix value that is $n!$

The next (last) difference is of course Zero.

We also note that all the followings derivative y^i holds the same property of the first derivative, so can be squared with a Sum or a Step Sum as shown so far, of course with the right Gnomon.

This are (updated from first version of this Vol.1) tables of the Recursive Difference, where is possible to see that the behavior of the Last Significative derivative is No longer the same of the continuous known derivative:

x	x^2	M4(x)	Delta2	Delta 3	Delta 4	Delta5	Delta6	Delta7	Delta8	Delta9	Delta10	Delta11	Delta12	Delta13
0	0	0	0	0	0									
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	4	3	2	1	0	-1	-2	-3	-4	-5	-6	-7	-8	-9
3	9	5	2	0	-1	-1	0	2	5	9	14	20	27	35
4	16	7	2	0	0	1	2	2	0	-5	-14	-28	-48	-75
5	25	9	2	0	0	0	-1	-3	-5	-5	0	14	42	90
6	36	11	2	0	0	0	0	1	4	9	14	14	0	-42
7	49	13	2	0	0	0	0	0	-1	-5	-14	-28	-42	-42
8	64	15	2	0	0	0	0	0	0	1	6	20	48	90
9	81	17	2	0	0	0	0	0	0	0	-1	-7	-27	-75
10	100	19	2	0	0	0	0	0	0	0	0	1	8	35
11	121	21	2	0	0	0	0	0	0	0	0	0	-1	-9
12	144	23	2	0	0	0	0	0	0	0	0	0	0	1
13	169	25	2	0	0	0	0	0	0	0	0	0	0	0

x	x^3	M4(x)	Delta2	Delta 3	Delta 4	Delta5	Delta6	Delta7	Delta8	Delta9	Delta10	Delta11	Delta12	Delta13
0	0	0	0	0	0									
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	8	7	6	5	4	3	2	1	0	-1	-2	-3	-4	-5
3	27	19	12	6	1	-3	-6	-8	-9	-9	-8	-6	-3	1
4	64	37	18	6	0	-1	2	8	16	25	34	42	48	51
5	125	61	24	6	0	0	1	-1	-9	-25	-50	-84	-126	-174
6	216	91	30	6	0	0	0	-1	0	9	34	84	168	294
7	343	127	36	6	0	0	0	0	1	1	-8	-42	-126	-294
8	512	169	42	6	0	0	0	0	0	-1	-2	6	48	174
9	729	217	48	6	0	0	0	0	0	0	1	3	-3	-51
10	1000	271	54	6	0	0	0	0	0	0	0	-1	-4	-1
11	1331	331	60	6	0	0	0	0	0	0	0	0	1	5
12	1728	397	66	6	0	0	0	0	0	0	0	0	0	-1
13	2197	469	72	6	0	0	0	0	0	0	0	0	0	0

x	x^4	M4(x)	Delta2	Delta 3	Delta 4	Delta5	Delta6	Delta7	Delta8	Delta9	Delta10	Delta11	Delta12	Delta13
0	0	0	0	0	0									
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	16	15	14	13	12	11	10	9	8	7	6	5	4	3
3	81	65	50	36	23	11	0	-10	-19	-27	-34	-40	-45	-49
4	256	175	110	60	24	1	-10	-10	0	19	46	80	120	165
5	625	369	194	84	24	0	-1	9	19	19	0	-46	-126	-246
6	1296	671	302	108	24	0	0	1	-8	-27	-46	-46	0	126
7	2401	1105	434	132	24	0	0	0	-1	7	34	80	126	126
8	4096	1695	590	156	24	0	0	0	0	1	-6	-40	-120	-246
9	6561	2465	770	180	24	0	0	0	0	0	-1	5	45	165
10	10000	3439	974	204	24	0	0	0	0	0	0	1	-4	-49
11	14641	4641	1202	228	24	0	0	0	0	0	0	0	-1	3
12	20736	6095	1454	252	24	0	0	0	0	0	0	0	0	1
13	28561	7825	1730	276	24	0	0	0	0	0	0	0	0	0

Recursive differences take us from x^n to $n!$ and over...

As you can see in this tables is also possible to continue the table After the Last Difference (that is $n!$) on the Right with what I'll call The Ghost Nexus Numbers, and the Ghost Composite Develop, with no limits, suggesting a sort of non symmetric anti-binomial develop (that is for math as a sort of anti-matter, for what I don't know there is a real physical relation).

$n!$ as Sum of $(n+1)$ Power Terms coming from a trick on the Binomial Develop:

From the previous table, so from the Recursive Difference Property we can see that in general is true that collecting $n+1$ Following Real Numbers build as $(p, p-1, p-2, ..., p-n)$ we can always have back the value of $n!$ using the Tartaglia's triangle (so binomial coefficients) and the following simple exchange of variable:
For any $p \in \mathbb{R}$, if $n = 2$ for example:

$$p^2 - 2(p-1)^2 + (p-2)^2$$

$$= p^2 - 2(p^2 - 2p + 1) + (p^2 - 4p + 4)$$

$$= p^2 - (2p^2 - 4p + 2) + (p^2 - 4p + 4)$$

$$= p^2 + (-2p^2 + 4p - 2) + (p^2 - 4p + 4)$$

$$= (p^2 - 2p^2 + p^2) + (4p - 4p) + (-2 + 4) = 1 * 2 = 2!$$

For $n = 3$

$$p^3 - 3(p-1)^3 + 3(p-2)^{3}\check{}\,(p-3)^3$$

$$= p^3 - 3(p^3 - 3p^2 + 3p - 1) + 3(p^3 - 6p^2 + 12p - 8) - (p^3 - 9p^2 + 27p - 27)$$

$$= p^3 - (3p^3 - 9p^2 + 9p - 3) + (3p^3 - 18p^2 + 36p - 24) - (p^3 - 9p^2 + 27p - 27)$$

$$= p^3 + (-3p^3 + 9p^2 - 9p + 3) + (3A^3 - 18p^2 + 36p - 24) + (-p^3 + 9p^{2}\check{}\,27p + 27)$$

$$= (p^3 - 3p^3 + 3p^3 - p^3) + (9p^2 - 18p^2 + 9p^2) + (-9p + 36p - 27A) + (3 - 24 + 27) = 6 = 1*2*3 = 3!$$

etc... Or using the Known Notation for any $p \in \mathbb{R}$:

$$n! = \sum_{k=0}^{n} \left(-1^k \binom{n}{k} (p-k)^n \right)$$

that it's true that for any $p \in \mathbb{N}$:

$$n! = \sum_{k=0}^{n} \left((-1)^k \binom{n}{k} \sum_{X=1}^{p-k} [X^n - (X-1)^n] \right)$$

and for any $p \in \mathbb{Q}$ and for some $p \in \mathbb{R}$ under condition we have seen (remembering that the Integer K is the one let $p * K \in \mathbb{N}$, or the Irrational depending by a known Irrational (typically an n-th root of an integer), has nothing to do with the integer k of the n over k notation):

$$n! = \sum_{k=0}^{n} \left((-1)^k \binom{n}{k} \sum_{x=1/K}^{p-k} M_{n,K} \right)$$

A^n as Sum of $(A-1)^n$ and the following Integer derivative:

For a $A > n$ we can have A^n with the Sum of all the Integer Difference $\Delta_{1,2,3,\ldots n;A-1}$ calculate for the row $A-1$

For example:

$$A^3 = (A-1)^3 + \Delta_1 + \Delta_2 + \Delta_3|_{A-1} = [(A-1)^3] + [(3(A-1)^2 - 3(A-1) + 1] + [6(A-2)] + 6$$

So if $A = 5$; $n = 3$ we have:

$$A^3 = [(A-1)^3] + [(3(A-1)^2 - 3(A-1)+1] + [6(A-2)] + 6 = [(5-1)^3] + [(3*(5-1)^2 - 3*(5-1)+1] + [6*(5-2)] + 6 = 125$$

The fact that we need to work with $A > n$ is due to the construction of the Telescoping Sum Triangle: we have zero as difference, or different value from $n!$ in the first rows of such triangles.
Nevertheless it is possible to have the right Sum also for little value of A, so for $A < n$ in some special case, but just if $n = Even$. Hereafter some example:
So if $A = 3$; $n = 4$ we have:

$$3^4 = (A-1)^4 + \Delta_1 + \Delta_2 + \Delta_3 + \Delta_3|_{(A-1)=2} = 16 + 15 + 14 + 13 + 12 + 11 = 81$$

This is not true for example for $A = 3$ and $n = 3$, in fact:

$$3^3 = 27 \neq (A-1)^3 + \Delta_1 + \Delta_2 + \Delta_3 + \Delta_3|_{(A-1)=2} = 8 + 7 + 6 + 5 = 26$$

and:

$$3^3 = 27 \neq (A-1)^3 + \Delta_1 + \Delta_2 + \Delta_3 + \Delta_3|_{(A-1)=2} = 8 + 7 + 6 + 5 + 4 = 30$$

A^n in Binary Form as Sum of $n!$ and 1:

From the previous chapters we note, for the property of the Telescoping Sum that Power of Integers can be written in a Binary Form, so as Sum of $n!$ and 1:

$$A^n = \left\lfloor \frac{X^n}{n!} \right\rfloor + \left\lceil \frac{X^n}{n!} \right\rceil = X^n \, mod \, n! + Rest$$

Then one can felt into mistakes that we have a new method to identify n-th Power via their Class of Rest, but unfortunately, this is False, since in a finite interval of numbers there are no number enough to describe an infinite set of other numbers, And I think can be true just in case we make a test for the Rest are in the form 2^m we can always find in the case $n = 2$.

X	X^2	FLOOR	CEIL	X	X^3	FLOOR	CEIL	X	X^4	FLOOR	CEIL
1	1	0	1	1	1	0	1	1	1	0	1
2	4	2	2	2	8	1	7	2	16	0	16
3	9	4	5	3	27	4	23	3	81	3	78
4	16	8	8	4	64	10	54	4	256	10	246
5	25	12	13	5	125	20	105	5	625	26	599
6	36	18	18	6	216	36	180	6	1296	54	1242
7	49	24	25	7	343	57	286	7	2401	100	2301
8	64	32	32	8	512	85	427	8	4096	170	3926
9	81	40	41	9	729	121	608	9	6561	273	6288
10	100	50	50	10	1000	166	834	10	10000	416	9584
11	121	60	61	11	1331	221	1110	11	14641	610	14031
12	144	72	72	12	1728	288	1440	12	20736	864	19872
13	169	84	85	13	2197	366	1831	13	28561	1190	27371
14	196	98	98	14	2744	457	2287	14	38416	1600	36816
15	225	112	113	15	3375	562	2813	15	50625	2109	48516
16	256	128	128	16	4096	682	3414	16	65536	2730	62806
17	289	144	145	17	4913	818	4095	17	83521	3480	80041
18	324	162	162	18	5832	972	4860	18	104976	4374	100602
19	361	180	181	19	6859	1143	5716	19	130321	5430	124891
20	400	200	200	20	8000	1333	6667	20	160000	6666	153334
21	441	220	221	21	9261	1543	7718	21	194481	8103	186378
22	484	242	242	22	10648	1774	8874	22	234256	9760	224496
23	529	264	265	23	12167	2027	10140	23	279841	11660	268181
24	576	288	288	24	13824	2304	11520	24	331776	13824	317952
25	625	312	313	25	15625	2604	13021	25	390625	16276	374349
26	676	338	338	26	17576	2929	14647	26	456976	19040	437936
27	729	364	365	27	19683	3280	16403	27	531441	22143	509298
28	784	392	392	28	21952	3658	18294	28	614656	25610	589046
29	841	420	421	29	24389	4064	20325	29	707281	29470	677811
30	900	450	450	30	27000	4500	22500	30	810000	33750	776250
31	961	480	481	31	29791	4965	24826	31	923521	38480	885041
32	1024	512	512	32	32768	5461	27307	32	1048576	43690	1004886
33	1089	544	545	33	35937	5989	29948	33	1185921	49413	1136508
34	1156	578	578	34	39304	6550	32754	34	1336336	55680	1280656
35	1225	612	613	35	42875	7145	35730	35	1500625	62526	1438099
36	1296	648	648	36	46656	7776	38880	36	1679616	69984	1609632

X	X^5	FLOOR	CEIL	X	X^6	FLOOR	CEIL	X	X^7	FLOOR	CEIL
1	1	0	1	1	1	0	1	1	1	0	1
2	32	0	32	2	64	0	64	2	128	0	128
3	243	2	241	3	729	1	728	3	2187	0	2187
4	1024	8	1016	4	4096	5	4091	4	16384	3	16381
5	3125	26	3099	5	15625	21	15604	5	78125	15	78110
6	7776	64	7712	6	46656	64	46592	6	279936	55	279881
7	16807	140	16667	7	117649	163	117486	7	823543	163	823380
8	32768	273	32495	8	262144	364	261780	8	2097152	416	2096736
9	59049	492	58557	9	531441	738	530703	9	4782969	949	4782020
10	100000	833	99167	10	1000000	1388	998612	10	10000000	1984	9998016
11	161051	1342	159709	11	1771561	2460	1769101	11	19487171	3866	19483305
12	248832	2073	246759	12	2985984	4147	2981837	12	35831808	7109	35824699
13	371293	3094	368199	13	4826809	6703	4820106	13	62748517	12450	62736067
14	537824	4481	533343	14	7529536	10457	7519079	14	105413504	20915	105392589
15	759375	6328	753047	15	11390625	15820	11374805	15	170859375	33900	170825475
16	1048576	8738	1039838	16	16777216	23301	16753915	16	268435456	53261	268382195
17	1419857	11832	1408025	17	24137569	33524	24104045	17	410338673	81416	410257257
18	1889568	15746	1873822	18	34012224	47239	33964985	18	612220032	121472	612098560
19	2476099	20634	2455465	19	47045881	65341	46980540	19	893871739	177355	893694384
20	3200000	26666	3173334	20	64000000	88888	63911112	20	1280000000	253968	1279746032
21	4084101	34034	4050067	21	85766121	119119	85647002	21	1801088541	357358	1800731183
22	5153632	42946	5110686	22	113379904	157472	113222432	22	2494357888	494912	2493862976
23	6436343	53636	6382707	23	148035889	205605	147830284	23	3404825447	675560	3404149887
24	7962624	66355	7896269	24	191102976	265420	190837556	24	4586471424	910014	4585561410
25	9765625	81380	9684245	25	244140625	339084	243801541	25	6103515625	1211015	6102304610
26	11881376	99011	11782365	26	308915776	429049	308486727	26	8031810176	1593613	8030216563
27	14348907	119574	14229333	27	387420489	538084	386882405	27	1.046E+10	2075466	10458277737
28	17210368	143419	17066949	28	481890304	669292	481221012	28	1.3493E+10	2677168	13490251344
29	20511149	170926	20340223	29	594823321	826143	593997178	29	1.725E+10	3422594	17246453715
30	24300000	202500	24097500	30	729000000	1012500	727987500	30	2.187E+10	4339285	21865660715
31	28629151	238576	28390575	31	887503681	1232644	886271037	31	2.7513E+10	5458852	27507155259
32	33554432	279620	33274812	32	1073741824	1491308	1072250516	32	3.436E+10	6817408	34352920960
33	39135393	326128	38809265	33	1291467969	1793705	1289674264	33	4.2618E+10	8456040	42609986937
34	45435424	378628	45056796	34	1544804416	2145561	1542658855	34	5.2523E+10	10421299	52512928845
35	52521875	437682	52084193	35	1838265625	2553146	1835712479	35	6.4339E+10	12765733	64326531142
36	60466176	503884	59962292	36	2176782336	3023308	2173759028	36	7.8364E+10	15548445	78348615651

Unfortunately, as you can see not all the Rest are different, as we can espect since $n!$ is wide, but not enough to include the Set of all the Integers...

The question for the reader is: are just 2^{2m} the problems ?

Table 13: The Rest Modulo $n!$ for X^2, X^3, X^4

X	X^2	**FLOOR**	**REST**	**X**	X^3	**FLOOR**	**REST**	**X**	X^4	**FLOOR**	**REST**
1	1	0	1	1	1	0	1	1	1	0	1
2	4	2	2	2	8	1	7	2	16	0	16
3	9	4	5	3	27	4	23	3	81	3	78
4	16	8	8	4	64	10	54	4	256	10	246
5	25	12	13	5	125	20	105	5	625	26	599
6	36	18	18	6	216	36	180	6	1296	54	1242
7	49	24	25	7	343	57	286	7	2401	100	2301
8	64	32	**32**	8	512	85	427	8	4096	170	3926
9	81	40	41	9	729	121	608	9	6561	273	6288
10	100	50	50	10	1000	166	834	10	10000	416	9584
11	121	60	61	11	1331	221	1110	11	14641	610	14031
12	144	72	72	12	1728	288	1440	12	20736	864	19872
13	169	84	85	13	2197	366	1831	13	28561	1190	27371
14	196	98	98	14	2744	457	2287	14	38416	1600	36816
15	225	112	113	15	3375	562	2813	15	50625	2109	48516
16	256	128	**128**	16	4096	682	3414	16	65536	2730	62806
17	289	144	145	17	4913	818	4095	17	83521	3480	80041
18	324	162	162	18	5832	972	4860	18	104976	4374	100602
19	361	180	181	19	6859	1143	5716	19	130321	5430	124891
20	400	200	200	20	8000	1333	6667	20	160000	6666	153334
21	441	220	221	21	9261	1543	7718	21	194481	8103	186378
22	484	242	242	22	10648	1774	8874	22	234256	9760	224496
23	529	264	265	23	12167	2027	10140	23	279841	11660	268181
24	576	288	288	24	13824	2304	11520	24	331776	13824	317952
25	625	312	313	25	15625	2604	13021	25	390625	16276	374349
26	676	338	338	26	17576	2929	14647	26	456976	19040	437936
27	729	364	365	27	19683	3280	16403	27	531441	22143	509298
28	784	392	392	28	21952	3658	18294	28	614656	25610	589046
29	841	420	421	29	24389	4064	20325	29	707281	29470	677811
30	900	450	450	30	27000	4500	22500	30	810000	33750	776250
31	961	480	481	31	29791	4965	24826	31	923521	38480	885041
32	1024	512	512	32	32768	5461	27307	32	1048576	43690	1004886
33	1089	544	545	33	35937	5989	29948	33	1185921	49413	1136508
34	1156	578	578	34	39304	6550	32754	34	1336336	55680	1280656
35	1225	612	613	35	42875	7145	35730	35	1500625	62526	1438099
36	1296	648	648	36	46656	7776	38880	36	1679616	69984	1609632

Table 14: The Rest Modulo $n!$ for X^5, X^6

X	X^5	**FLOOR**	**REST**	**X**	X^6	**FLOOR**	**REST**
1	1	0	1	1	1	0	1
2	32	0	**32**	2	64	0	64
3	243	2	241	3	729	1	728
4	1024	8	1016	4	4096	5	4091
5	3125	26	3099	5	15625	21	15604
6	7776	64	7712	6	46656	64	46592
7	16807	140	16667	7	117649	163	117486
8	32768	273	32495	8	262144	364	261780
9	59049	492	58557	9	531441	738	530703
10	100000	833	99167	10	1000000	1388	998612
11	161051	1342	159709	11	1771561	2460	1769101
12	248832	2073	246759	12	2985984	4147	2981837
13	371293	3094	368199	13	4826809	6703	4820106
14	537824	4481	533343	14	7529536	10457	7519079
15	759375	6328	753047	15	11390625	15820	11374805
16	1048576	8738	1039838	16	16777216	23301	16753915
17	1419857	11832	1408025	17	24137569	33524	24104045
18	1889568	15746	1873822	18	34012224	47239	33964985
19	2476099	20634	2455465	19	47045881	65341	46980540
20	3200000	26666	3173334	20	64000000	88888	63911112
21	4084101	34034	4050067	21	85766121	119119	85647002
22	5153632	42946	5110686	22	113379904	157472	113222432
23	6436343	53636	6382707	23	148035889	205605	147830284
24	7962624	66355	7896269	24	191102976	265420	190837556
25	9765625	81380	9684245	25	244140625	339084	243801541
26	11881376	99011	11782365	26	308915776	429049	308486727
27	14348907	119574	14229333	27	387420489	538084	386882405
28	17210368	143419	17066949	28	481890304	669292	481221012
29	20511149	170926	20340223	29	594823321	826143	593997178
30	24300000	202500	24097500	30	729000000	1012500	727987500
31	28629151	238576	28390575	31	887503681	1232644	886271037
32	33554432	279620	33274812	32	1073741824	1491308	1072250516
33	39135393	326128	38809265	33	1291467969	1793705	1289674264
34	45435424	378628	45056796	34	1544804416	2145561	1542658855
35	52521875	437682	52084193	35	1838265625	2553146	1835712479
36	60466176	503884	59962292	36	2176782336	3023308	2173759028

Chapt.14: The Integer Derivative Formulas, for $Y = X^n$ curve

The Integer derivative function is not the same of the Classic One, in fact as we can immediately see that on the table:

Relation between Following DELTA, respect to the CLASSIC DERIVATIVE					
X	Delta 1 = 3x^2-3x+1	d/dx (Delta 1) = 6x-3	Delta 2 = Delta1(x)-Delta1(x-1)	d/dx (Delta 2) = 6	Delta 3 = Delta2(x)-Delta2(x-1)
0	0	0	0	0	0
1	1	3	1	6	1
2	7	9	6	6	6
3	19	15	12	6	6
4	37	21	18	6	6
5	61	27	24	6	6

We cannot apply the classic derivation Rule to pass from the integer First derivative Delta1 ($\delta 1$) to to the Second derivative Delta2 ($\delta 2$) using the Known Derivation Rules. So:

$\delta 1 = 3X^2 - 3X + 1$ and $\frac{d}{dx}(\delta 1) = 6x - 3 \neq \delta 2$

So we need to investigate in such difference to better understand how the Integer derivative behave respect to X

Starting to see, as an example, what happens for $n = 3$ we can see that for the First Difference (the initial difference, me and Mr. Nexus, forgot to consider) we cannot apply the Same Rule of the Rest of the Column. The number of the initial "strange" difference, seems not obeying at any distribution law, Depends on 3 variables:

- the n-th Degree of $Y = X^n$ we are considering

- the n-th Delta we are considering ($\delta 1$, $\delta 2$, $\delta 3$... etc...)

- the n-th Row we are considering (X=1, X=2.... etc....

So for example for $n = 3$:

$\delta 1 = 3X^2 - 3X + 1$ (do not depends by other factors than X)

$\delta 2_1 = 1$ - $\delta 2_{2>..} = 6(X - 1)$

And for Delta3:

$\delta 3_1 = 1$ - $\delta 3_1 = 5$ - $\delta 3_{3>..} = 6$

As we can see the things becomes more complicate rising n since there will be more Integer Derivative Initial Gnomons:
So for example for $n = 4$:

X	Delta 1 = 4x^3-6x^2+4x-1	d/dx (Delta 1) = 12x^2-12x+4	Delta 2 = Delta1(x)-Delta1(x-1)	d/dx (Delta 2) = 24x-12	Delta 3 = Delta2(x)-Delta2(x-1)	d/dx (Delta 3) = 24	Delta 4 = Delta3(x)-Delta3(x-1)
0	0	0	0	0	0	0	0
1	1	4	1	12	1	24	1
2	15	28	14	36	13	24	12
3	65	76	50	60	36	24	23
4	175	148	110	84	60	24	24
5	369	244	194	108	84	24	24
6	671	364	302	132	108	24	24
7	1105	508	434	156	132	24	24
8	1695	676	590	180	156	24	24
9	2465	868	770	204	180	24	24
10	3439	1084	974	228	204	24	24
11	4641	1324	1202	252	228	24	24

$\delta 1 = 4X^3 - 6X^2 + 4X - 1$ (do not depend by other factors than X)

$\delta 2_1 = 1$ - $\delta 2_{2>..} = 12(X-1)^2 + 2$

And for Delta3:

$\delta 3_1 = 1$ - $\delta 3_1 = 5$ - $\delta 3_{3>..} = 6$

For the complete sequence see: http://oeis.org/A101104

Chapt.15: The Ghost Nexus Numbers, and the Ghost Composite Develop

What is interesting is that the new "information", once read by Columns, becomes the coefficient of a New Composite Develop:

In case n= 3 they are the equals of the Newton's like develop for a "composite" power:

Delta5:
1 3 -3 -1
are the coefficient for:

$$x^3 + 3x^2 - 3x - 1 = 0$$

or:

$$(x-1) * (x^2 + 4x + 1) = 0$$

In case n= Even they are the equals of the Newtons develop for what we can call a "Composite Non Perfect Power":

For example n=4

Delta7
1 9 -10 -10 9 1
is:

$$x^5 + 9x^4 - 10x^3 - 10x^2 + 9x + 1 = 0$$

or:

$$(x-1)^2 * (x+1) * (x^2 + 10x + 1) = 0$$

Interesting is that this is in relation with:

Delta6

1 10 0 -10 -1
that is:

$$x^4 + 10x^4 - 10x - 1 = 0$$

or

$$(x-1)(x+1)(x^2 + 10x + 1) = 0$$

And of course also with

Delta8

1 8 -19 0 19 -8 -1
or:

$$(x-1)^3(x+1)(x^2+10x+1)=0$$

.... and infinite more new ones we have no time here to better investigate nor define...

The Last Linear Integer derivative

What is also interesting is to see that the equation for The Last Linear Integer derivative y^L is:

$$y^L = n!x - n!/2$$

This, of course will have a big relevance into Fermat the last Theorem proof and in other similar problems.

Chapt.16: Lebesgue Integer/Rational Integration via Sum and Step Sum

All I did for my modified Riemann's Sum till the Integral can be adjusted for what I call my Lebesgue's (Like) Sum and Integral.

Remembering Lebesgue Sum can uses Longitudinal Bars instead of Vertical ones (plus other properties that at the moment we will not investigate), we have just to search for the proper Horizontal Gnomon I'll call $M_{n,y}$.

If you remember I've already said that once we use Rational for $n > 2$ we lose invertible property, so there is no way to find a non recursive dependence of the new $Y - Gnomons$ from X and so the new Gnomons function $M_{n,y}$ is no longer a monotone rising function: that means that for $n > 2$ the Gnomons function M_n we saw so far, is not invertible.

Taking a look to the picture, remembering what we said in the previous chapters, it's clear we can square the area below the derivative, in the Lebesgue's direction, using a recursive path for the Height of the new $M_{n,y}$ Gnomon:

$$A^n = \sum_{x=1}^{A}((M_{(n)|X} - M_{(n)|X1})(A + 1 - X))$$

where:

$M_{(n)|x}$ - is the well known Complicate Integer Modulus calculated for each single value of x

$M_{(n)|x-1}$ - is the Complicate Integer Modulus calculated for the single value of $x - 1$

The Lebesgue Integer / Rational Sum can also work in the Rational REMEMBERING to put $x = X/k$:

$$A^n = \sum_{x=1/K}^{A}((M_{(n,k)|x} - M_{(n,k)|x-1})(AK + 1 - Kx))$$

where:

$M_{(n,k)|x}$ - is the well known Complicate Rational Modulus (K dependent) calculated for the single value of x

$M_{(n,k)|x-1}$ - is the well known Complicate Rational Modulus (K dependent) calculated for the single value of $x - 1$

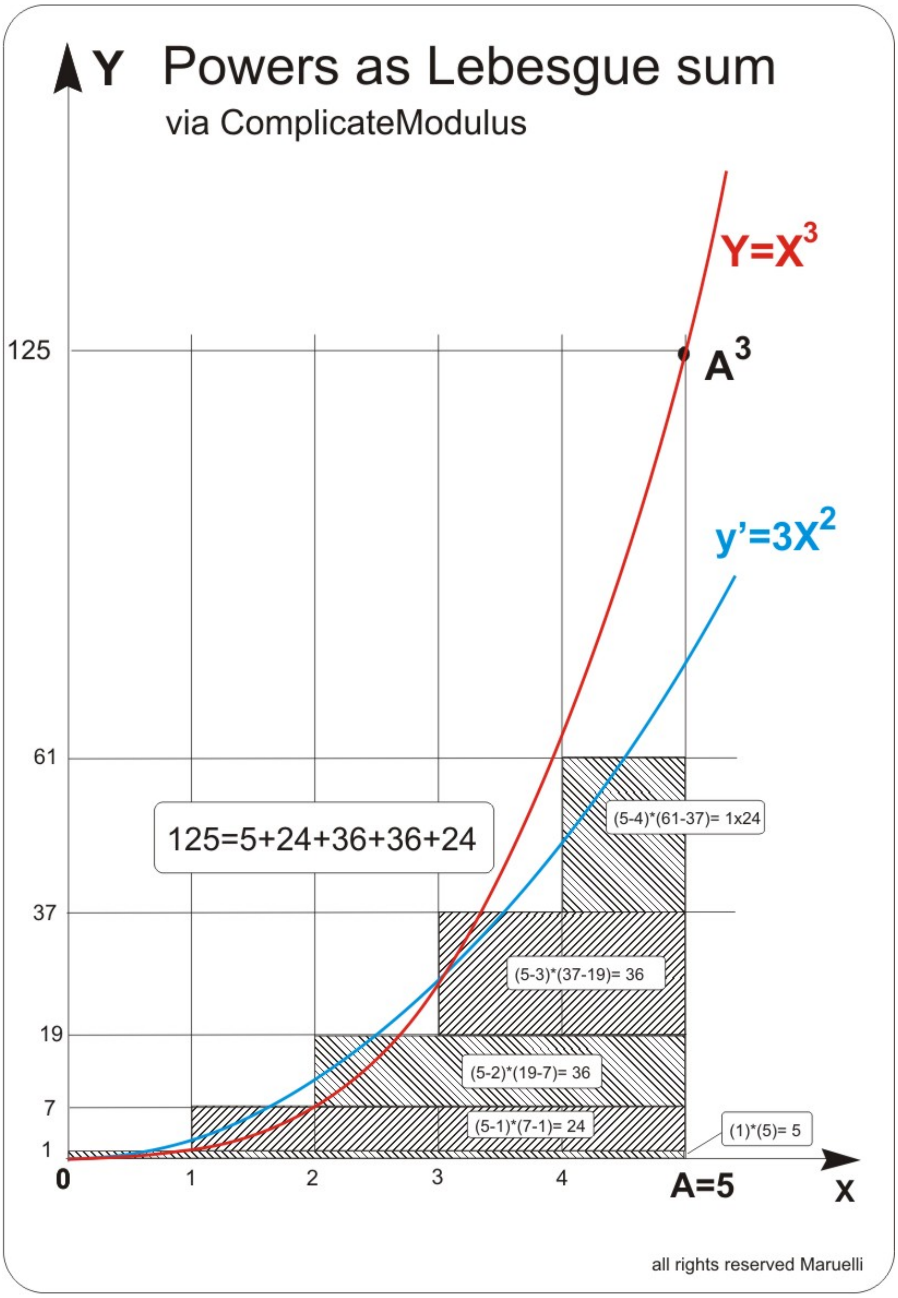
And of course is possible to go to the limit, having the Integral:

$$A^n = \lim_{K\to\infty} \sum_{x=1/K}^{A}((M_{(n,k)|x} - M_{(n,k)|x-1})(AK - 1 + Kx)) = \int_0^A nx^{n-1}dx$$

Here in the graph what happens in the integers for $n = 3$...

As we can see we do not square using a linear Delta2y, where Delta2y is the difference between two following Gnomons $M_{n|x}$ and $M_{n|x-1}$

I hope will not soo hard to prove there is no other way to do that... since the Integer / Rational derivative is a non invertible function.



Here an example of Lebesgue Integer Integration for n=3

$$A^3 = \sum_{x=1}^{A} ((3x^2 - 3x + 1)|_x - (3x^2 - 3x + 1)|_{x-1})(A + 1 - x))$$

We can immediately see that just in case $n = 2$, with a linear derivative we have (If we exclude the first Gnomon) Both LINEAR M_n and $M_{n,y}$ Gnomons

The concerning on this let me found a new family of triangles for A^n. you can find two examples on: www.oeis.org

`http://oeis.org/A276158`

`http://oeis.org/A276189`

Here an example for $n = 2$, $n = 3$ and $n = 4$:

Maruelli-Lebesgue TRIANGLE **n=2**

A	a_0	a_1	a_2	S3	S4	S5	6	7	8	9	10	A^2
1	1											1
2	2	2										4
3	3	4	2									9
4	4	6	4	2								16
5	5	8	6	4	2							25
6	6	10	8	6	4	2						36
7	7	12	10	8	6	4	2					49
8	8	14	12	10	8	6	4	2				64
9	9	16	14	12	10	8	6	4	2			81
10	10	18	16	14	12	10	8	6	4	2		100
11	11	20	18	16	14	12	10	8	6	4	2	121

Maruelli-Lebesgue TRIANGLE **n=3**

A	a_0	a_1	a_2	S3	S4	S5	6	7	8	9	10	A^3
1	1											1
2	2	6										8
3	3	12	12									27
4	4	18	24	18								64
5	5	24	36	36	24							125
6	6	30	48	54	48	30						216
7	7	36	60	72	72	60	36					343
8	8	42	72	90	96	90	72	42				512
9	9	48	84	108	120	120	108	84	48			729
10	10	54	96	126	144	150	144	126	96	54		1000
11	11	60	108	144	168	180	180	168	144	108	60	1331

Maruelli-Lebesgue TRIANGLE **n=4**

A	a_0	a_1	a_2	S3	S4	S5	6	7	8	9	10	A^4
1	1											1
2	2	14										16
3	3	28	50									81
4	4	42	100	110								256
5	5	56	150	220	194							625
6	6	70	200	330	388	302						1296
7	7	84	250	440	582	604	434					2401
8	8	98	300	550	776	906	868	590				4096
9	9	112	350	660	776	1208	1302	920	770			6561
10	10	126	400	770	970	1510	1736	1250	1540	974		10000
11	11	140	450	880	1164	1812	2170	1580	2310	1948	1202	14641

Powers as a Lebesgue Sum of integers. For Example $2^3 = 8 = 2 + 6$; $3^3 = 8 = 3 + 12 + 12$ etc...

Here the A276158 sequence with the generating formula:

A276158 Triangle read by rows: T(n,k) = 6*k*(n + 1 - k) for 0 < k <= n; for k = 0, T(n,0) = n + 1. 1

1, 2, 6, 3, 12, 12, 4, 18, 24, 18, 5, 24, 36, 36, 24, 6, 30, 48, 54, 48, 30, 7, 36, 60, 72, 72, 60, 36, 8, 42, 72, 90, 96, 90, 72, 42, 9, 48, 84, 108, 120, 120, 108, 84, 48, 10, 54, 96, 126, 144, 150, 144, 126, 96, 54 (list; table; graph; refs; listen; history; text; internal format)

OFFSET 0,2

COMMENTS The row sums of the triangle provide the positive terms of A000578.
Similar triangles can be generated by the formula P(n,k,m) = (Q(k+1,m)-Q(k,m))*(n+1-k), where Q(i,r) = i^r-(i-1)^r, 0 < k <= n, and P(n,0,m) = n+1. T(n,k) is the case m=3, that is T(n,k) = P(n,k,3).

LINKS Table of n, a(n) for n=0..54.

FORMULA Sum_{k=0..n} T(n,k) = T(n,0)^3 = A000578(n+1).
G.f. as triangle: (1+4*x*y + x^2*y^2)/((1-x)^2*(1-x*y)^2). - Robert Israel, Aug 31 2016
T(n,n-h) = (h+1)*A008458(n-h) for 0 <= h <= n. Therefore, the main diagonal of the triangle is A008458. - Bruno Berselli, Aug 31 2016

EXAMPLE Triangle starts:

```
------------------------------------------------
n \ k |  0   1    2    3    4    5    6    7
------------------------------------------------
0     |  1;
1     |  2,  6;
2     |  3, 12,  12;
3     |  4, 18,  24,  18;
4     |  5, 24,  36,  36,  24;
5     |  6, 30,  48,  54,  48,  30;
6     |  7, 36,  60,  72,  72,  60,  36;
7     |  8, 42,  72,  90,  96,  90,  72,  42;
...
```

MAPLE

```
T:= (n, k) -> `if`(k=0, n+1, 6*k*(n+1-k)):
seq(seq(T(n, k), k=0..n), n=0..30); # Robert Israel, Aug 31 2016
```

MATHEMATICA

```
Table[If[k == 0, n + 1, 6 k (n + 1 - k)], {n, 0, 10}, {k, 0, n}] //
  Flatten (* Michael De Vlieger, Aug 25 2016 *)
```

PROG

```
(PARI) T(n, k) = if (k==0, n+1, 6*k*(n+1-k));
tabl(nn) = for (n=0, nn, for (k=0, n, print1(T(n, k), ", "));
  print(); \\ Michel Marcus, Aug 25 2016
(MAGMA) [IsZero(k) select n+1 else 6*k*(n+1-k): k in [0..n], n in
  [0..10]]; // Bruno Berselli, Aug 31 2016
(MAGMA) /* As triangle (see the second comment): */ m:=3;
  Q:=func<i, r | i^r-(i-1)^r>; P:=func<n, k, m | IsZero(k) select
  n+1 else (Q(k+1, m)-Q(k, m))*(n+1-k)>; [[P(n, k, m): k in
  [0..n]]: n in [0..10]]; // Bruno Berselli, Aug 31 2016
```

CROSSREFS Cf. A000578, A008458, A276189.
Sequence in context: A064736 A243618 A063929 * A092393 A207901 A054619
Adjacent sequences: A276155 A276156 A276157 * A276159 A276160 A276161

KEYWORD nonn,tabl

AUTHOR Stefano Maruelli, Aug 22 2016

To have $a(m)^n$ just sum all the terms of a line. For example to have 7^3:
$7^3 = 7 + 36 + 60 + 72 + 72 + 60 + 36 = 343$

The general formula for the M_n, y Gnomons is:

$$M_n, y = (M_n|_x - M_n|_{x-1})(A + 1 - x))$$

So the general Power of an integer A can be written as:

$$A^n = \sum_{x=1}^{A} (M_n|_x - M_n|_{x-1})(A + 1 - x))$$

We can immediately see that just in case $n = 2$, with a linear derivative we have Both LINEAR M_n and $M_{n,y}$ Gnomons, so in this case only, we can invert the Integer / Rational derivative.

As shown for the Nexus Numbers, also here the table can be continued, showing its Symmetrical Behavior:

Maruelli-Lebesgue TRIANGLE			n=2										
A	a_0	a_1	a_2	S3	S4	S5	6	7	8	9	10		A^2
0	0	-2	-4	-6	-8	-10	-12	-14	-16	-18	-20		
1	1	0	-2	-4	-6	-8	-10	-12	-14	-16	-18		1
2	2	2	0	-2	-4	-6	-8	-10	-12	-14	-16		4
3	3	4	2	0	-2	-4	-6	-8	-10	-12	-14		9
4	4	6	4	2	0	-2	-4	-6	-8	-10	-12		16
5	5	8	6	4	2	0	-2	-4	-6	-8	-10		25
6	6	10	8	6	4	2	0	-2	-4	-6	-8		36
7	7	12	10	8	6	4	2	0	-2	-4	-6		49
8	8	14	12	10	8	6	4	2	0	-2	-4		64
9	9	16	14	12	10	8	6	4	2	0	-2		81
10	10	18	16	14	12	10	8	6	4	2	0		100
11	11	20	18	16	14	12	10	8	6	4	2		121
12	12	22	20	18	16	14	12	10	8	6	4		144

Maruelli-Lebesgue TRIANGLE			n=3										
A	a_0	a_1	a_2	S3	S4	S5	6	7	8	9	10		A^3
0	0	-6	-24	-54	-96	-150	-216	-294	-384	-486	-600		
1	1	0	-12	-36	-72	-120	-180	-252	-336	-432	-540		1
2	2	6	0	-18	-48	-90	-144	-210	-288	-378	-480		8
3	3	12	12	0	-24	-60	-108	-168	-240	-324	-420		27
4	4	18	24	18	0	-30	-72	-126	-192	-270	-360		64
5	5	24	36	36	24	0	-36	-84	-144	-216	-300		125
6	6	30	48	54	48	30	0	-42	-96	-162	-240		216
7	7	36	60	72	72	60	36	0	-48	-108	-180		343
8	8	42	72	90	96	90	72	42	0	-54	-120		512
9	9	48	84	108	120	120	108	84	48	0	-60		729
10	10	54	96	126	144	150	144	126	96	54	0		1000
11	11	60	108	144	168	180	180	168	144	108	60	0	1331

While it is clear that for a Line the Balancing Point is the same of the Medium Point both on x and in y: $BP_i = MP_i = (x_{i,1/2}, y_{i,1/2}$

We will see in the Vol.2 what happen if we try to fix different conditions (like Fermat's one) on a derivative that is a Line, or a Curve ($n > 2$).
I'll follow 2 ways: both will look in how BP is geometrically fixed, the first one involve simple concerning on the relative position of BP respect to known things: the Medium (or Center) Point MP, the second one will show that we can pack X_m
We can immediately make some concerning on:

- we know from Telescoping Sum Property that for any derivative (also the following) the Exceeding Area A^+ will equate the Missing one A^-, without going out of Integer numbers and Proportional Areas,

- but once we ask how much the value of such areas is the only way to calculate them is to go infinitesimal and make the integral.

And this is due to what I call the Infinite Descent, that is not what Fermat discover so what is actually known under that name, but there is no way to better call this infinite process of approaching to an existing limit:

if we try to change the scale of the picture zooming in, we will see that still if we continuous to zoom in, so we keep X_{i+1} closer and closer to X_i, the condition that fix X_m rest an inequality that told us just $r > q$. But since we know X_m exist and it can be rise at the limit after infinite zooming in, than we prove X_m for all the curved derivative is an Irrational Value.

This is in fact the process known as Dedekind Cut. It sound like an Axiom, but it is now well proved.

All the work of the Vol.2 will be dedicated to problems involving powers, and the new way offered by this new method of investigating powers via Complicate Modulus.

Chapt. 17: What PARTIAL SUMs are

We have not all the knowledge to produce a New more general type of Summations I've called **PARTIAL SUMs** or **Magic Sums** since the results will often be an unpredictable surprise.
Partial Sums (here defined in their first basic version) are defined as:
Sums of values coming from a CARRIER FUNCTION, calculated for a value called MODULATOR.
The $CarrierFunction$ can be a continuous function, the $MODULATOR$ is the set I of the point on witch we calculate the Modulator Function, so the internal Terms of the Sum.

As shown in the previous chapters, I extent the concept of a classic SUM with Integers Index, as much as possible, to work with Rational and Irrationals too, but there is a big freedom in creating the elements that will builds the set of the Index $\mathbb{I}$.

I hope you try with on a simple .XLS file how powerful will be this "new" instrument.

The basic concept is to create a Sum of the type:

$$A = \sum_{x \in \mathbb{I}} f(x)$$

Where:

- The elements of the set $\mathbb{I}$ are the $x \in \mathbb{D}$, where $\mathbb{D}$ is a Definite Domain, be it under the previous rules $\mathbb{N}$, $\mathbb{Q}$or $\mathbb{R}$, or one of their Sub Set, and it is used instead of the classic integer Index i

- The set $\mathbb{I}$ is build using a known Function, called **MODULATOR** here for example $x = sin(x/10)$

- The internal term in the Sum is called: **CARRIER** and will be calculated at the values coming from the Modulator.

- The result will be very interesting because sometimes unexpected.

PARTIAL SUM produces MAGIC EFFECTS:

I present here just few example of how strange will be the result of this PARTIAL SUM, suggesting you to create your own example and also try to imagine the connection for chaos theory and Qbit behavior...

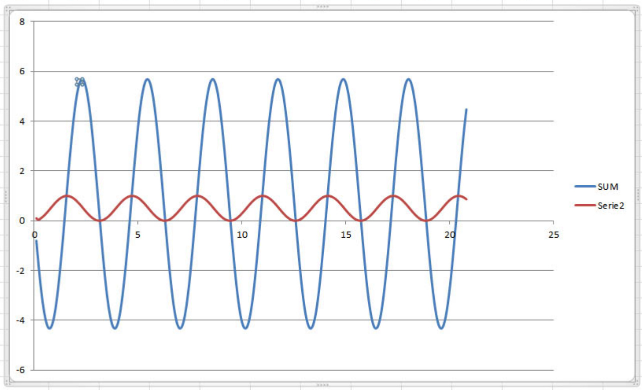
Example 1:

As CARRIER here we put the "Gnomon" $(2x-1)$ (I remember it is the Gnomon of the Square function $y = x^2$)

As MODULATOR here we keep: $x = sin(x/10)$, where $x = 1, 2, 3.....x \in \mathbb{N}^+$

So we will see what will happen if we Sum the value coming from the sin of a Rational angles: 0.1, 0.2,....x/10.

X		x = sin(X)	F(x) = 2x-1	SUM
1	0.1	0.099833417	-0.800333167	-0.80033
2	0.2	0.039469503	-0.921060994	-1.72139
3	0.3	0.087332193	-0.825335615	-2.54673
4	0.4	0.151646645	-0.696706709	-3.24344
5	0.5	0.229848847	-0.540302306	-3.78374
6	0.6	0.318821123	-0.362357754	-4.1461
7	0.7	0.415016429	-0.169967143	-4.31606
8	0.8	0.514599761	0.029199522	-4.28686
9	0.9	0.613601047	0.227202095	-4.05966
10	1	0.708073418	0.416146837	-3.64352
11	1.1	0.794250559	0.588501117	-3.05501
12	1.2	0.868696858	0.737393716	-2.31762
13	1.3	0.928444377	0.856888753	-1.46073
14	1.4	0.97111117	0.942222341	-0.51851
15	1.5	0.994996248	0.989992497	0.471483
16	1.6	0.999147388	0.998294776	1.469778
17	1.7	0.983399096	0.966798193	2.436576
18	1.8	0.948379208	0.896758416	3.333335
19	1.9	0.895483856	0.790967712	4.124302
20	2	0.82682181	0.653643621	4.777946
21	2.1	0.745130411	0.490260821	5.268207
22	2.2	0.653666435	0.30733287	5.57554
23	2.3	0.556076263	0.112152527	5.687692
24	2.4	0.456250508	-0.087498983	5.600193
25	2.5	0.358168907	-0.283662185	5.316531
26	2.6	0.265741664	-0.468516671	4.848014
27	2.7	0.182653562	-0.634692876	4.213321
28	2.8	0.112217061	-0.775565879	3.437756
29	2.9	0.057240242	-0.885519517	2.552236
30	3	0.019914857	-0.960170287	1.592066
31	3.1	0.001728951	-0.996542097	0.595524
32	3.2	0.003407541	-0.993184919	-0.39766



Partial SUM: Modulator: sin(X/10)
Carrier: (2x-1)

(c) Stefano Maruelli

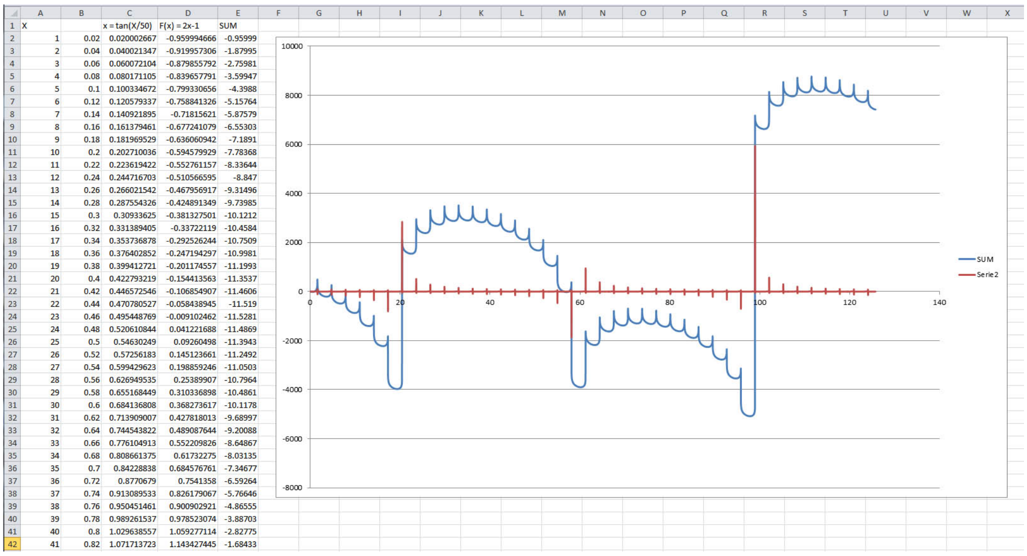
In the picture the (interpolated) Carrier in Red and in Blue the (interpolated) result of the Sum.
One can expect that the result of the Sum will diverge but, vice versa it holds bounded.

Example 2:

CARRIER $= 2x - 1$

MODULATOR $=$ a collection of Rational Value given simply dividing by 50 the set of $x \in \mathbb{N}^+$, than applying the TAN function.

Again not so predictable graph will appears.



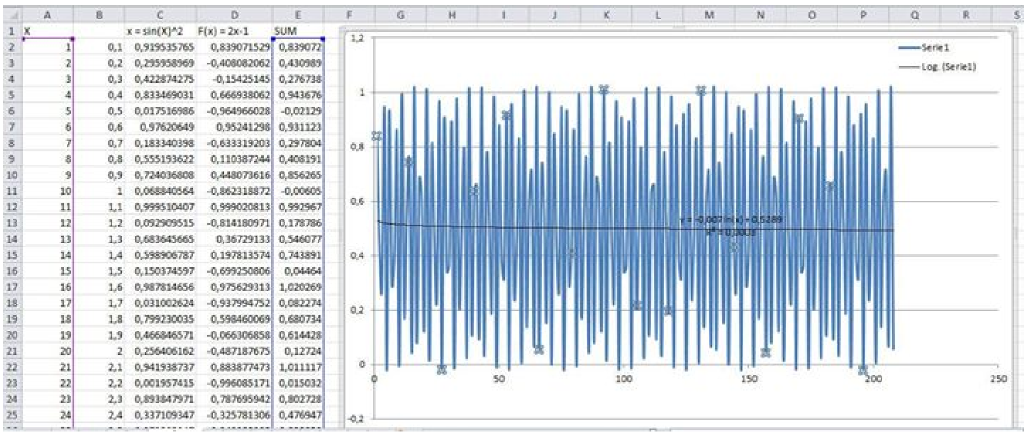
	A	B	C	D	E
1	X		x = tan(X/50)	F(x) = 2x-1	SUM
2	1	0.02	0.020002667	-0.959994666	-0.95999
3	2	0.04	0.040021347	-0.919957306	-1.87995
4	3	0.06	0.060072104	-0.879855792	-2.75981
5	4	0.08	0.080171105	-0.839657791	-3.59947
6	5	0.1	0.100334672	-0.799330656	-4.3988
7	6	0.12	0.120579337	-0.758841326	-5.15764
8	7	0.14	0.140921895	-0.71815621	-5.87579
9	8	0.16	0.161379461	-0.677241079	-6.55303
10	9	0.18	0.181969529	-0.636060942	-7.1891
11	10	0.2	0.202710036	-0.594579929	-7.78368
12	11	0.22	0.223619422	-0.552761157	-8.33644
13	12	0.24	0.244716703	-0.510566595	-8.847
14	13	0.26	0.266021542	-0.467956917	-9.31496
15	14	0.28	0.287554326	-0.424891349	-9.73985
16	15	0.3	0.30933625	-0.381327501	-10.1212
17	16	0.32	0.331389405	-0.33722119	-10.4584
18	17	0.34	0.353736878	-0.292526244	-10.7509
19	18	0.36	0.376402852	-0.247194297	-10.9981
20	19	0.38	0.399412721	-0.201174557	-11.1993
21	20	0.4	0.422793219	-0.154413563	-11.3537
22	21	0.42	0.446572546	-0.106854907	-11.4606
23	22	0.44	0.470780527	-0.058438945	-11.519
24	23	0.46	0.495448769	-0.009102462	-11.5281
25	24	0.48	0.520610844	0.041221688	-11.4869
26	25	0.5	0.54630249	0.09260498	-11.3943
27	26	0.52	0.57256183	0.145123661	-11.2492
28	27	0.54	0.599429623	0.198859246	-11.0503
29	28	0.56	0.626949535	0.25389907	-10.7964
30	29	0.58	0.655168449	0.310336898	-10.4861
31	30	0.6	0.684136808	0.368273617	-10.1178
32	31	0.62	0.713909007	0.427818013	-9.68997
33	32	0.64	0.744543822	0.489087644	-9.20088
34	33	0.66	0.776104913	0.552209826	-8.64867
35	34	0.68	0.808661375	0.61732275	-8.03135
36	35	0.7	0.84228838	0.684576761	-7.34677
37	36	0.72	0.8770679	0.7541358	-6.59264
38	37	0.74	0.913089533	0.826179067	-5.76646
39	38	0.76	0.950451461	0.900902921	-4.86555
40	39	0.78	0.989261537	0.978523074	-3.88703
41	40	0.8	1.029638557	1.059277114	-2.82775
42	41	0.82	1.071713723	1.143427445	-1.68433

Example 3:

CARRIER $= 2x - 1$

MODULATOR $=$ a collection of Rational Value $x = x/10$ with $x \in \mathbb{N}^+$,

than applying the function: $SIN(x)^2$ function.



	A	B	C	D	E
1	X		x = sin(X)^2	F(x) = 2x-1	SUM
2	1	0,1	0,919535765	0,839071529	0,839072
3	2	0,2	0,295958969	-0,408082062	0,430989
4	3	0,3	0,422874275	-0,15425145	0,276738
5	4	0,4	0,833469031	0,666938062	0,943676
6	5	0,5	0,017516986	-0,964966028	-0,02129
7	6	0,6	0,97620649	0,95241298	0,931123
8	7	0,7	0,183340398	-0,633319203	0,297804
9	8	0,8	0,555193622	0,110387244	0,408191
10	9	0,9	0,724036808	0,448073616	0,856265
11	10	1	0,068840564	-0,862318872	-0,00605
12	11	1,1	0,999510407	0,999020813	0,992967
13	12	1,2	0,092909515	-0,814180971	0,178786
14	13	1,3	0,683645665	0,36729133	0,546077
15	14	1,4	0,598906787	0,197813574	0,743891
16	15	1,5	0,150374597	-0,699250806	0,04464
17	16	1,6	0,987814656	0,975629313	1,020269
18	17	1,7	0,031002624	-0,937994752	0,082274
19	18	1,8	0,799230035	0,598460069	0,680734
20	19	1,9	0,466846571	-0,066306858	0,614428
21	20	2	0,256406162	-0,487187675	0,12724
22	21	2,1	0,941938737	0,883877473	1,011117
23	22	2,2	0,001957415	-0,996085171	0,015032
24	23	2,3	0,893847971	0,787695942	0,802728
25	24	2,4	0,337109347	-0,325781306	0,476947

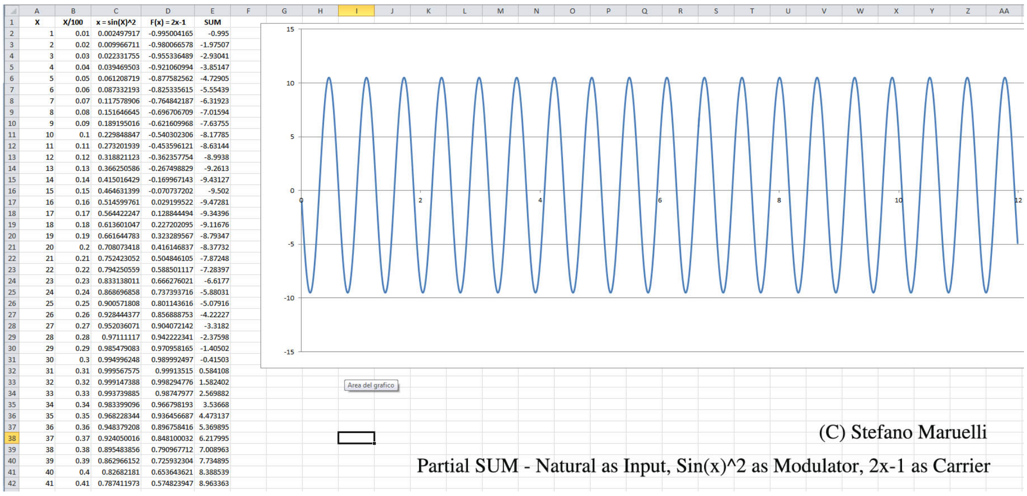
Example 4:

CARRIER $= (2x - 1)$

MODULATOR $=$ a collection of Rational Value $x = x/100$ with $x \in \mathbb{N}^{+}$,

than applying the function: $SIN(x)^2$

Here seems that with the interpolation of the value don't change lot the known SIN behavior.

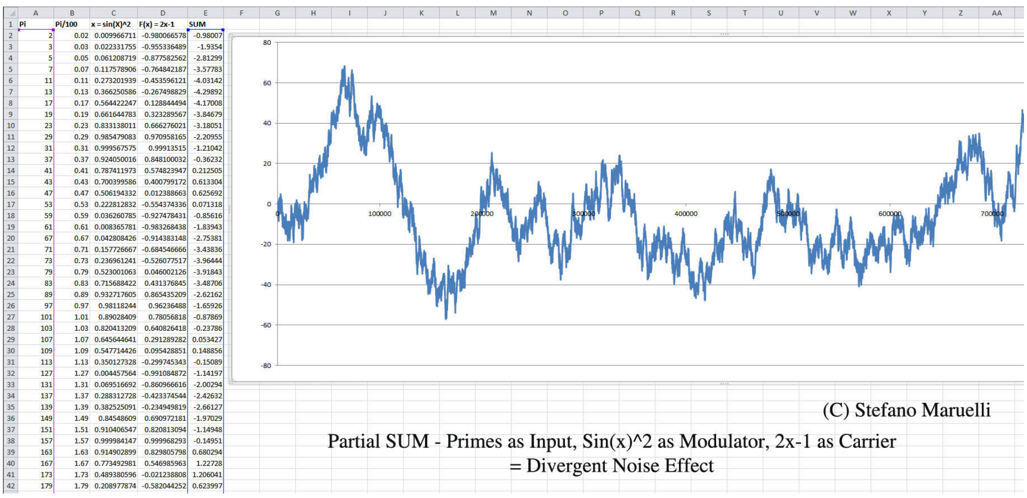


Example 5:

CARRIER $= (2x - 1)$

MODULATOR $=$ a collection of Rational Value given simply dividing by 100 each Prime Numbers only, so $x = \pi_(i)/100$

than applying the function: $SIN(x)^2$. "Random" or "Noise" behavior here is well expected.

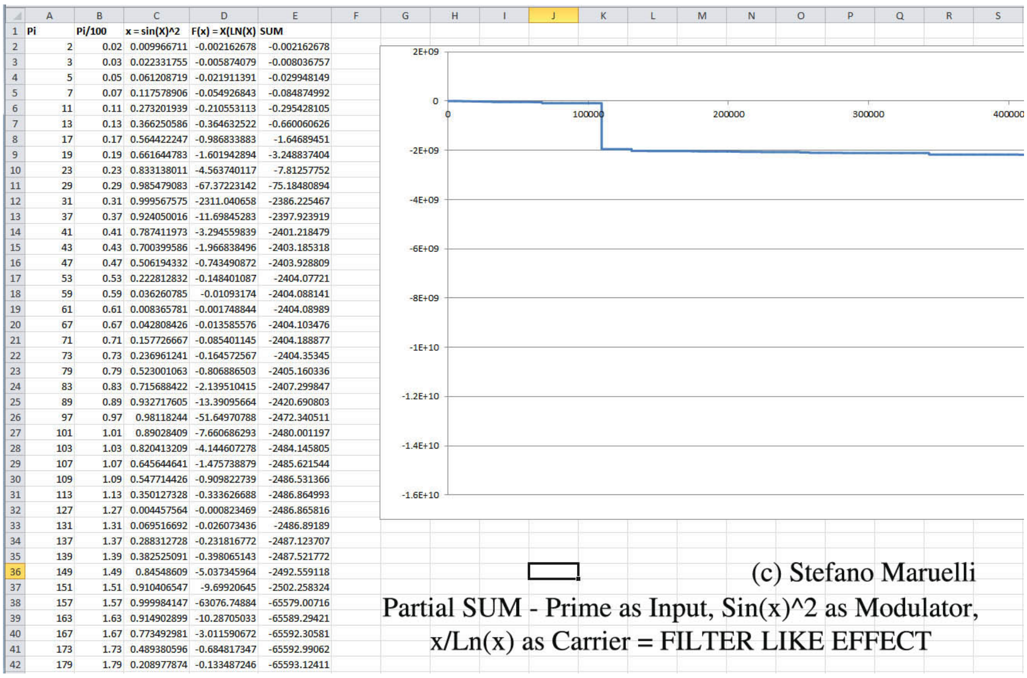


Example 6:

CARRIER = $(2x - 1)$

MODULATOR = is a collection of Rational Value given simply dividing by 100 the set of Prime Numbers only, $x = \pi_(i)/100$

than applying the function: $x = (xLn(x))$. A sort of Filter effect is shown.



	A	B	C	D	E
1	Pi	Pi/100	x = sin(X)^2	F(x) = X(LN(X)	SUM
2	2	0.02	0.009966711	-0.002162678	-0.002162678
3	3	0.03	0.022331755	-0.005874079	-0.008036757
4	5	0.05	0.061208719	-0.021911391	-0.029948149
5	7	0.07	0.117578906	-0.054926843	-0.084874992
6	11	0.11	0.273201939	-0.210553113	-0.295428105
7	13	0.13	0.366250586	-0.364632522	-0.660060626
8	17	0.17	0.564422247	-0.986833883	-1.64689451
9	19	0.19	0.661644783	-1.601942894	-3.248837404
10	23	0.23	0.833138011	-4.563740117	-7.81257752
11	29	0.29	0.985479083	-67.37223142	-75.18480894
12	31	0.31	0.999567575	-2311.040658	-2386.225467
13	37	0.37	0.924050016	-11.69845283	-2397.923919
14	41	0.41	0.787411973	-3.294559839	-2401.218479
15	43	0.43	0.700399586	-1.966838496	-2403.185318
16	47	0.47	0.506194332	-0.743490872	-2403.928809
17	53	0.53	0.222812832	-0.148401087	-2404.07721
18	59	0.59	0.036260785	-0.01093174	-2404.088141
19	61	0.61	0.008365781	-0.001748844	-2404.08989
20	67	0.67	0.042808426	-0.013585576	-2404.103476
21	71	0.71	0.157726667	-0.085401145	-2404.188877
22	73	0.73	0.236961241	-0.164572567	-2404.35345
23	79	0.79	0.523001063	-0.806886503	-2405.160336
24	83	0.83	0.715688422	-2.139510415	-2407.299847
25	89	0.89	0.932717605	-13.39095664	-2420.690803
26	97	0.97	0.98118244	-51.64970788	-2472.340511
27	101	1.01	0.89028409	-7.660686293	-2480.001197
28	103	1.03	0.820413209	-4.144607278	-2484.145805
29	107	1.07	0.645644641	-1.475738879	-2485.621544
30	109	1.09	0.547714426	-0.909822739	-2486.531366
31	113	1.13	0.350127328	-0.333626688	-2486.864993
32	127	1.27	0.004457564	-0.000823469	-2486.865816
33	131	1.31	0.069516692	-0.026073436	-2486.89189
34	137	1.37	0.288312728	-0.231816772	-2487.123707
35	139	1.39	0.382525091	-0.398065143	-2487.521772
36	149	1.49	0.84548609	-5.037345964	-2492.559118
37	151	1.51	0.910406547	-9.69920645	-2502.258324
38	157	1.57	0.999984147	-63076.74884	-65579.00716
39	163	1.63	0.914902899	-10.28705033	-65589.29421
40	167	1.67	0.773492981	-3.011590672	-65592.30581
41	173	1.73	0.489380596	-0.684817347	-65592.99062
42	179	1.79	0.208977874	-0.133487246	-65593.12411

(c) Stefano Maruelli

Partial SUM - Prime as Input, Sin(x)^2 as Modulator, x/Ln(x) as Carrier = FILTER LIKE EFFECT

As you can see the Square Gnomon Carrier $(2x - 1)$ produce lot of interesting behavior once in relation with trigonometric functions we know depends on the square function. I hope all this will be interesting for those are studying the Qbit behavior since it can probably give some information on how to find information in what actually is supposed chaos.

Chapt. 18: Squaring Known Curves with Gnomons:

Gnomons can be used to square, with approximation, several curves.

All is born considering the telescoping sum property for $Y = X^n$ derivative, and in that case it don't care if you stretch the scale of x, it means how far is 1 from 0 in the x, respect to how far is 1 form 0 in y.

As shown in the first chapters, nothing change in the result of the Summation (till the integral) if you move step 1 or step $1/K$ or dx, don't care how wide are Gnomons and for so you can make the variable exchange to $x = X/K$ to square the curve $Y' = nX^{n-1}$ with the width you prefer / need.

While in general for other curves this will not happen and the Integral is littlest / bigger than the Gnomons area depending witch value we choose for the Integer / Rational Height. Here I present few example to show how interesting is this research for math and physics since it show some interesting result of different ways to measuring the same unmeasurable things.

1) Gnomons over Hyperbole

If you try to cover the 1st quadrant Hyperbole's area, so the area bellow $Y = 1/X$ curve, with Gnomons given by $Y = 1/\lfloor x \rfloor$, you can see that Gnomon's Area is bigger than the one bellow the $Y = 1/X$ curve, due to the obvious fact that it is a continuous sinking function that do not have the same properties of the Parabolas.

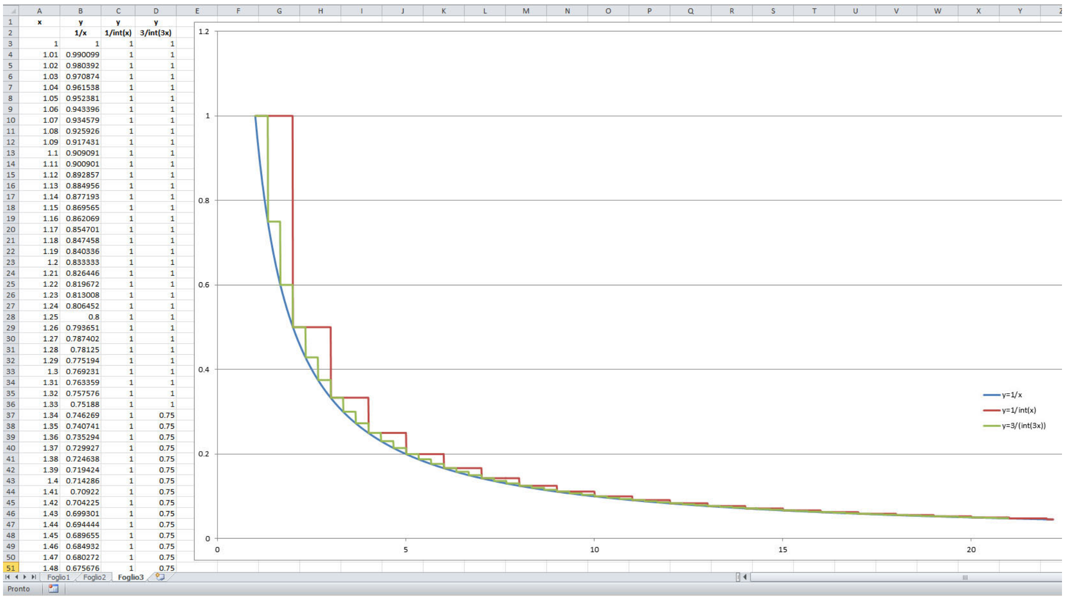
If we keep the *Base* of the Gnomons equal to 1, the difference between the area bellow the curve and the one of the Gnomons is the well known Euler-Mascheroni constant:

$$\gamma = \lim_{n\to\infty} \left(-\ln n + \sum_{k=1}^{n} \frac{1}{k} \right) = \int_1^\infty \left(\frac{1}{\lfloor x \rfloor} - \frac{1}{x} \right) dx. \tag{3}$$

Now we make here the process I've already shown for parabolas, to pass from Sum to Integral, so you will change $x = X/K$ in the Sum (adjusting the limit as shown in my trick), than you will see that the area bellow your Gnomons decreases rising K, till you've at the limit for $K \to \infty$ exactly the area bellow $Y = 1/X$ curve.

So in other terms $Y = 1/X$ derivative is characterized by $\gamma_* = 0$

Where γ_* is a new more general value (and not just Eulero's constant) representing the goodness of the approximation.



Rising K the area of the Gnomons go closer and closer to the one of $y = 1/x$. In the example K=1 (the difference between the areas correspond to the known γ) and K=3 (that produce a new γ_* or γ_{*K}), and for $K \to \infty$ the two areas are equal and $\gamma_\infty = 0$).

So γ_* is a new toy, and several, more deep, concerning will follows.

Once again we can chose Upper Gnomons (as in the example), or Lower Gnomons and see that at the limit for $K \to \infty$ are both equal.

What is already known and clear is that the precision of the measure depends on the **Right precision** of the instrument we use, so depends on witch K we choose, but a bigger K is not an insurance of a better measure.

This will be more clear if we compare the result of the PC error just in case we Sum exactly the areas of the Gnomons given by the Integral, having for example base a fraction of e. Still if e is an approximated value, than the Integral is not precise too, cutting it in several columns of are equal to the approximated integral area, the Sum is not affected by a significant error in term of PC digits: in fact the result is the same for the whole integral, and for the whole sum of columns, independently from how many (1, 10, 10 or 1000) are them. Probably to see some significant digit we have to rise lot the number of columns, while for the approximated Rational Gnomons the error rises/fells lot.

Here one example using K=10, 100, 1000. No difference from $Log(P)$ and the Sum of 10,

CHECK FOR DIFFERENT PRECISION IN MEASURING THE INTEGRAL AS INTEGRAL, OR SUM OF COLUMNS COMING FROM THE INTEGRAL

K=1/100

X	p = 1+ x*e	Log (p+1)-log (p)	
0.01	1.027182818	0.011647746	
0.02	1.054365637	0.011343497	
0.03	1.081548455	0.011054738	
0.04	1.108731273	0.010780316	
0.05	1.135914091	0.01051919	
0.06	1.16309691	0.010270415	
0.07	1.190279728	0.010033135	
0.08	1.217462546	0.009806572	
0.09	1.244645365	0.009590016	
0.1	1.271828183	0.009382818	
0.11	1.299011001	0.009184385	
0.12	1.326193819	0.008994171	
0.13	1.353376638	0.008811676	
0.14	1.380559456	0.00863644	
0.63	2.712517552	0.004374128	
0.64	2.73970037	0.004330511	
0.65	2.766883188	0.004287756	
0.66	2.794066007	0.004245837	
0.67	2.821248825	0.00420473	
0.68	2.848431643	0.004164411	
0.69	2.875614462	0.004124857	
0.7	2.90279728	0.004086048	
0.71	2.929980098	0.004047963	
0.72	2.957162916	0.004010581	
0.73	2.984345735	0.003973883	LOG(P)
0.74	3.011528553	0.003937851	0.47878698518133900
1			
	SUM=	0.478786985181339000	NO DIFFERENCE

K=1/10

X	p = 1+ x*e	Log (p+1)-log (p)	
0.1	1.271828183	0.104428444	
0.2	1.543656366	0.084122184	
0.3	1.815484549	0.070441929	
0.4	2.087312731	0.060594965	
0.5	2.359140914	0.053166361	
0.6	2.630969097	0.047361864	
0.7	2.90279728	0.04270096	LOG(P)
0.8	3.174625463	0.038875788	0.50169249528209500
	SUM=	0.501692495282095000	NO DIFFERENCE

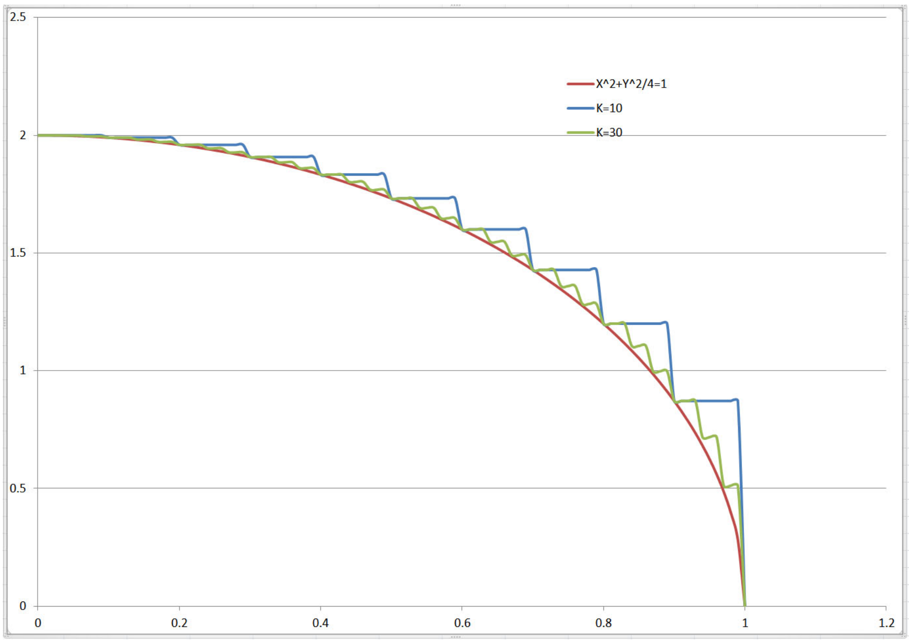
K=1/1000

X	p = 1+ x*e	Log (p+1)-log (p)	
0.001	1.002718282	0.001178933	
0.002	1.005436564	0.001175742	
0.003	1.008154845	0.001172567	
0.004	1.010873127	0.00116941	
0.005	1.013591409	0.001166269	
0.006	1.016309691	0.001163146	
0.007	1.019027973	0.001160039	
0.008	1.021746255	0.001156949	
0.009	1.024464536	0.001153875	
0.01	1.027182818	0.001150817	
0.011	1.0299011	0.001147776	
0.012	1.032619382	0.00114475	
0.013	1.035337664	0.001141741	
0.014	1.038055946	0.001138747	
0.063	1.171251755	0.001009097	
0.064	1.173970037	0.001006758	
0.065	1.176688319	0.00100443	
0.066	1.179406601	0.001002112	
0.067	1.182124883	0.000999805	
0.068	1.184843164	0.000997508	
0.069	1.187561446	0.000995223	
0.07	1.190279728	0.000992947	
0.071	1.19299801	0.000990682	
0.072	1.195716292	0.000988427	
0.073	1.198434573	0.000986183	
0.074	1.201152855	0.000983949	
0.075	1.203871137	0.000981724	
0.076	1.206589419	0.00097951	
0.077	1.209307701	0.000977306	
0.078	1.212025983	0.000975112	
0.079	1.214744264	0.000972927	
0.098	1.266391619	0.000933205	LOG(P)
0.099	1.269109901	0.000931205	0.10349923229073500
	SUM=	0.103499232290735000	NO DIFFERENCE
		(CTRL)	

100 or 1000 partial value of the integral. To have significant errors we have probably to rise lot K.

2) Gnomons over Ellipse

See the above integration via Step Sum of a quarter of an Ellipse:



Something of very interesting will happen rising K: while one expect that the precision of the measure will rise continuously, it is false due to the Sum of Approximation done by the PC so:

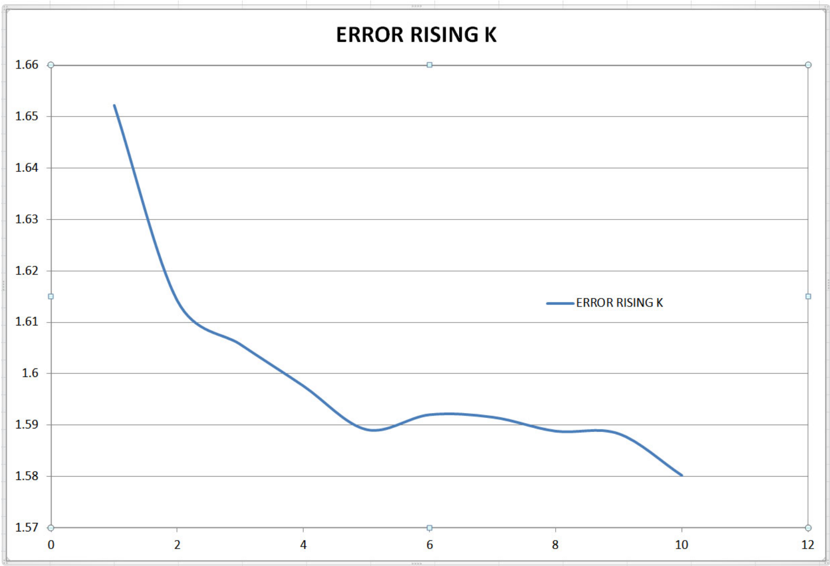
- From K=10 to $K = 20$ the error fells lot

- from K=20 to $K = 50$ is quasi linear but:

- $K = 50$ gives a better result than $K = 60$, and this is a very big problem for physics study. In this case 10 times more precise instrument gives better precision in the measure, but it is not always an insurance...

In the next page the Table with numbers and the Graph of the Error.

a=1	(4-4*(INT(K*x)/K)^2)^(1/2)									b=2
	K=10	K=20	K=30	K=40	K=50	K=60	K=70	K=80	K=90	K=100
X	T(10*x)/10)^2)^(1/2)	(4-4*(INT(30*x)/30)^2)^(1/2)		(4-4*(INT(50*x)/50)^2)^(1/2)						Y=(4-4*x^2)^(1/2)
0	2	2	2	2	2	2	2	2	2	2
0.01	2	2	2	2	2	2	2	2	2	1.999899997
0.02	2	2	2	2	1.99959996	1.9997222	1.9997959	1.9998437	1.9998765	1.99959996
0.03	2	2	2	1.999374902	1.99959996	1.9997222	1.9991835	1.9993749	1.9995061	1.999099797
0.04	2	2	1.99888858	1.999374902	1.998399359	1.9988886	1.9991835	1.9985933	1.9988886	1.998399359
0.05	2	1.997498436	1.99888858	1.997498436	1.998399359	1.9974984	1.9981624	1.9974984	1.9980237	1.997498436
0.06	2	1.997498436	1.99888858	1.997498436	1.996396754	1.9974984	1.996732	1.9974984	1.9969112	1.996396754
0.07	2	1.997498436	1.995550606	1.997498436	1.996396754	1.9955506	1.996732	1.9960899	1.9955506	1.995093983
0.08	2	1.997498436	1.995550606	1.994367068	1.993589727	1.9955506	1.9948914	1.9943671	1.9939414	1.993589727
0.09	2	1.997498436	1.995550606	1.994367068	1.993589727	1.9930435	1.9926395	1.992329	1.9920831	1.991883531
0.1	1.9899749	1.989974874	1.989974874	1.989974874	1.989974874	1.9899749	1.9899749	1.9899749	1.9899749	1.989974874
0.11	1.9899749	1.989974874	1.989974874	1.989974874	1.989974874	1.9899749	1.9899749	1.9899749	1.9899749	1.987863174
0.12	1.9899749	1.989974874	1.989974874	1.989974874	1.985547783	1.9863423	1.9868958	1.9873034	1.987616	1.985547783
0.75	1.4282857	1.322875656	1.359738537	1.322875656	1.345213738	1.3228757	1.3388999	1.3228757	1.3353688	1.322875656
0.76	1.4282857	1.322875656	1.359738537	1.322875656	1.299846145	1.3228757	1.3064987	1.3228757	1.3101692	1.299846145
0.77	1.4282857	1.322875656	1.284090686	1.322875656	1.299846145	1.2840907	1.3064987	1.2939764	1.2840907	1.276087771
0.78	1.4282857	1.322875656	1.284090686	1.263922466	1.251559028	1.2840907	1.2726319	1.2639225	1.2570787	1.251559028
0.79	1.4282857	1.322875656	1.284090686	1.263922466	1.251559028	1.2432037	1.2371791	1.2326293	1.2290717	1.226213684
0.8	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2
0.81	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.172859753
0.82	1.2	1.2	1.2	1.2	1.144727042	1.1542193	1.1609286	1.1659224	1.1697842	1.144727042
0.83	1.2	1.2	1.2	1.130265456	1.144727042	1.1542193	1.1197667	1.1302655	1.1383332	1.115526781
0.84	1.2	1.2	1.105541597	1.130265456	1.085172797	1.1055416	1.1197667	1.0928746	1.1055416	1.085172797
0.85	1.2	1.053565375	1.105541597	1.053565375	1.085172797	1.0535654	1.0762748	1.0535654	1.0712863	1.053565375
0.86	1.2	1.053565375	1.105541597	1.053565375	1.020588066	1.0535654	1.0301575	1.0535654	1.035422	1.020588066
0.87	1.2	1.053565375	0.997775303	1.053565375	1.020588066	0.9977753	1.0301575	1.0121141	0.9977753	0.986103443
0.88	1.2	1.053565375	0.997775303	0.968245837	0.949947367	0.9977753	0.9810448	0.9682458	0.9581361	0.949947367
0.89	1.2	1.053565375	0.997775303	0.968245837	0.949947367	0.9374907	0.9284615	0.9216154	0.9162457	0.911921049
0.9	0.8717798	0.871779789	0.871779789	0.871779789	0.871779789	0.8717798	0.8717798	0.8717798	0.8717798	0.871779789
0.91	0.8717798	0.871779789	0.871779789	0.871779789	0.871779789	0.8717798	0.8717798	0.8717798	0.8717798	0.829216498
0.92	0.8717798	0.871779789	0.871779789	0.871779789	0.783836718	0.7993053	0.8101398	0.8181534	0.8243216	0.783836718
0.93	0.8717798	0.871779789	0.871779789	0.759934208	0.783836718	0.7993053	0.7423075	0.7599342	0.7733206	0.735119038
0.94	0.8717798	0.871779789	0.718021974	0.759934208	0.682348884	0.718022	0.7423075	0.6959705	0.718022	0.682348884
0.95	0.8717798	0.6244998	0.718021974	0.6244998	0.682348884	0.6244998	0.6663945	0.6244998	0.6573422	0.6244998
0.96	0.8717798	0.6244998	0.718021974	0.6244998	0.56	0.6244998	0.5792324	0.6244998	0.5896222	0.56
0.97	0.8717798	0.6244998	0.512076383	0.6244998	0.56	0.5120764	0.5792324	0.5425634	0.5120764	0.486209831
0.98	0.8717798	0.6244998	0.512076383	0.444409721	0.397994975	0.5120764	0.4746642	0.4444097	0.4192881	0.397994975
0.99	0.8717798	0.6244998	0.512076383	0.444409721	0.397994975	0.3636237	0.3368522	0.315238	0.2973131	0.28213472
1	0	0	0	0	0	0	0	0	0	0
SUM * 0.0:	1.6522592	1.61423244	1.605725336	1.597625582	1.589134256	1.5920276	1.5915477	1.5888264	1.588323	1.580208516
ERROR:	0.0814628	0.043436113	1.605725336	0.026829255	0.018337929	0.0212313	0.0207513	0.0180301	0.0175267	0.009412189
TRUE AREA	1.5707963	pi.greco()*a*b /4								



Something of very interesting will happen rising K: while one expect that the precision of the measure will rise continuously, but as seen into the chart this is false, and I left the investigate on to understand why.

Is not hard to imagine why we define caos, or non deterministic, most of the nature events...

Chapt.19: The General n-th degree Equation Solving Algorithm:

To better understand the power of our Two Hand Clock, I'll present here a short example of how we can use it to solve also problems involving more than one Power, and or mixed terms, constant, so what we call Polynomial equations. Any Polynomial Equation of Any Degree.

It is proved true that a General Algorithm for solving ALL the n-th equations doesn't exist, due to the "Radical Closure", in other words there is no Radical Solution to any higher degree equation using the Known Multiplicative Algebra, but using my Additive Algebra, and understanding what a Radical is, in its general, most wide, definition, the problem vanish (but unfortunately it do not solve all the equations because some rest non reductible).

Living to someone else to debate if what I'll present is a Numerical Solution, or a more general concept for extract *Radicals* also from what is not a perfect power, just, so any Root of a Polynomial Equation, I'll present you here the simple methos to find the roots to Any Degree Equation (must be clear that sometimes works and sometimes not !).

A) The General Solving Method using CMA:

There is a General Solving Method for Polynomial equations using my CMA, it means that there is a More General Concept of what a Radical is: not just the result of the n.th root, coming from a special polynomial equation of the type: $X^n = const$, but any root coming from any n-th degree polynomial equation.

To extract Roots from Any Polynomial n-th Degree Equation you simply:

- Transform Any Power Term of the Polynomial Equation into a Sum from 1 to Rx if you're searching for integer solutions, or from $1/K$ to Rx if you're looking for Rational ones,

where Rx is ANY Root of such equation

Using the appropriate Complicate Real $M_{n,K}$ or Imaginary Modulus $M_{n,K,i}$ (in case the constant term is negative)

Then applying the Sum Properties you can Sum all the Terms of the Sums having back the solving (no always!) Polynomial (is our Integer or Rational derivative).

This Sum is the Algebraic representation of your equation, and sometimes exactly returns you any Integer or Rational Roots of the equation.

Imaginary Roots are not yet investigated enough at this time.

So with this new General Root Extractor (Algo, if it works) you will have back any of the Integer or Rational or Algebraic Irrational or imaginary Roots $Rx*$ you're looking for.

Where $|Rx*|$ is of course a Rational Number, but also at the Limit for $K \to \infty$ any Real Root (or zero) Rx coming from the Integral. So this Algo has no limits in its application, but of course the computation will truncate the numerical result of the Root in case it is an Irrational or a long Rational.

I start with the first most simple example of how to find the Roosts of a 2th degree equation:

$$X^2 - 5X + 6 = 0$$

1) We put:

$$X^2 = \sum_{1/K}^{Rx*}(2X - 1)$$

2) then we put:

$$5X = 5 * \sum_{1/K}^{Rx*}(1) = \sum_{1/K}^{Rx*}(5)$$

3) We can Sum all the terms with their sign under a single Sum, since the Upper Limit is the Same:

$$\sum_{1/K}^{Rx*}(2X - 6) = -6$$

4) So any Integer / Rational / Irrational, Positive or Negative Root of the Polynomial Equation $X^2 - 5X + 6 = 0$

will comes from the vice versa of the Sum, so as a Result of Recursive difference, so any time in the recursive difference we get a Rest equal to zero:

Roots of (X^2-5X+6=0)			
		Diff. from	
X	2X-6	6	
1	-4	2	
2	-2	0	first root
3	0	0	second root
4	2	2	

What is interesting here is that in some special case one Root is also the Root of First Rational derivative, infact for $X = 3$ we can have a root just in case the result of the single line computation is again zero so if:

$$(2X - 6) = -6$$

that of course has $X = 3$ as solution.

So as we well know we can write:

$$X^2 - 5X + 6 = 0- > (X - 2) * (X - 3) = 0$$

It is not hard to imagine why we define chaos, or non deterministic, most of the nature events...

How the trick works on Higher Degree Equation:

Here a more interesting example of the Most Famous Unsolvable 5th Degree Equation $X^5 - X + 1 = 0$

transforming each unknown in a Sum till the Unknown Upper Limit is the Integer or Rational or an Algebraic Irrational Root R_{x*}

close to the Real one is Rx, we can write, with precision $K = 10^m$:

$$X^5 - X = -1 \implies \sum_{1/K}^{Rx*} M_{5,K,i} - M_{1,K,i} = -1$$

We will have for so Two type of solutions:

- Integers and Rationals with a finite number of digits comes in the Computational Total Precision, so $R_{x*} = Rx$,

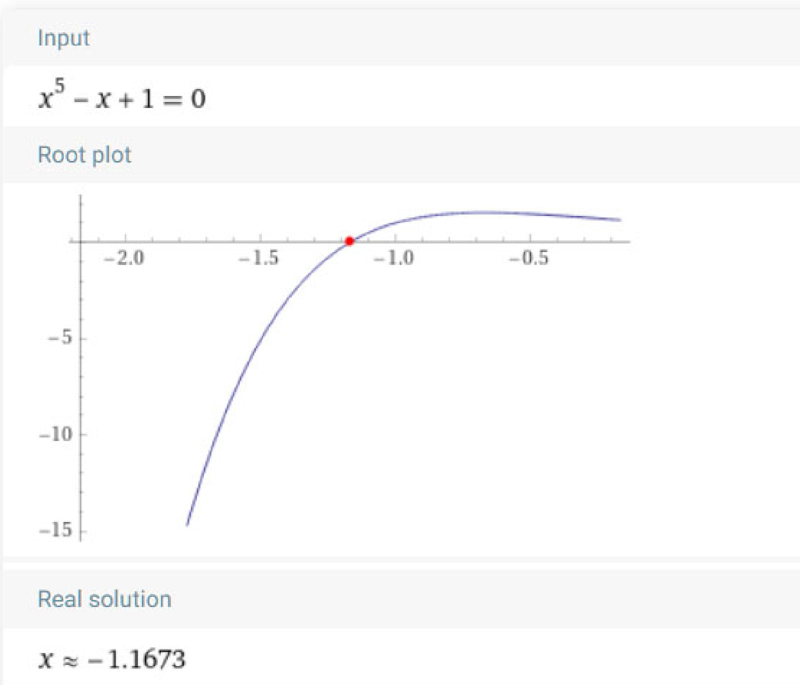
- while the other will be an approximation till the maximum precision we are able to raise with our computer, so with our maximum K, and $R_{x*} < Rx$.

In the above example we have defined a new Imaginary Specific Complicate Modulus, $M_{x^5-x,K,i}$ Algebraically solve this quintic equation:

$$x^5 - x = -1 \implies \lim_{K\to\infty} \sum_{1/K}^{Rx} M_{x^5-x,K,i} = -1$$

Of course the technical problem is that you must be capable to work with many significative digits, to have the desired precisions for the root.

Here is WOLFRAM solution:



And here is in the example keeping $K = 10^m = 10^4$ we get back exactly 4 correct digits (pls note that each digit you get rising m of 1, will be an exact decimal, one, not an approximated one).

Solution for X^5-X+1=0			r1= -1.1673039782614		r1*=1.1673	K=10000
		The Sum this part rise to r1^5		this rise to r1		Constant
X	x	5x^4/K-10x^3/K^2+10x^2/K^3-5x/K^4+1/K^5	Sum	1/K	difference:	-1
1	0,0001	-1E-25	-1E-25	-0,0001		-1,0001
2	0,0002	-3E-19	-3E-19	-0,0001		-1,0002
3	0,0003	-2,1E-18	-2,4E-18	-0,0001		-1,0003
4	0,0004	-7,8E-18	-1,02E-17	-0,0001		-1,0004
5	0,0005	-2,1E-17	-3,12E-17	-0,0001		-1,0005
6	0,0006	-4,65E-17	-7,77E-17	-0,0001		-1,0006
7	0,0007	-9,03E-17	-1,68E-16	-0,0001		-1,0007
11648	1,1648	-0,00092024	-2,144158124	-0,0001		-0,020641876
11649	1,1649	-0,000920556	-2,145078679	-0,0001		-0,019821321
11650	1,165	-0,000920872	-2,145999551	-0,0001		-0,019000449
11651	1,1651	-0,000921188	-2,146920739	-0,0001		-0,018179261
11652	1,1652	-0,000921504	-2,147842244	-0,0001		-0,017357756
11653	1,1653	-0,000921821	-2,148764064	-0,0001		-0,016535936
11654	1,1654	-0,000922137	-2,149686201	-0,0001		-0,015713799
11655	1,1655	-0,000922454	-2,150608655	-0,0001		-0,014891345
11656	1,1656	-0,00092277	-2,151531425	-0,0001		-0,014068575
11657	1,1657	-0,000923087	-2,152454513	-0,0001		-0,013245487
11658	1,1658	-0,000923404	-2,153377916	-0,0001		-0,012422084
11659	1,1659	-0,000923721	-2,154301637	-0,0001		-0,011598363
11660	1,166	-0,000924038	-2,155225675	-0,0001		-0,010774325
11661	1,1661	-0,000924355	-2,15615003	-0,0001		-0,00994997
11662	1,1662	-0,000924672	-2,157074702	-0,0001		-0,009125298
11663	1,1663	-0,000924989	-2,157999691	-0,0001		-0,008300309
11664	1,1664	-0,000925306	-2,158924997	-0,0001		-0,007475003
11665	1,1665	-0,000925624	-2,159850621	-0,0001		-0,006649379
11666	1,1666	-0,000925941	-2,160776562	-0,0001		-0,005823438
11667	1,1667	-0,000926259	-2,161702821	-0,0001		-0,004997179
11668	1,1668	-0,000926576	-2,162629398	-0,0001		-0,004170602
11669	1,1669	-0,000926894	-2,163556292	-0,0001		-0,003343708
11670	1,167	-0,000927212	-2,164483504	-0,0001		-0,002516496
11671	1,1671	-0,00092753	-2,165411033	-0,0001		-0,001688967
11672	1,1672	-0,000927848	-2,166338881	-0,0001		-0,000861119
11673	1,1673	-0,000928166	-2,167267047	-0,0001		-3,29532E-05
11674	1,1674	-0,000928484	-2,168195531	-0,0001		0,000795531
		the Root Extractor for this quintic is for so:				
		M_{X5-X,K,i}= 5x^4/K-10x^3/K^2+10x^2/K^3-5x/K^4				

The solution obey to this proven rule, is , theoretically due to the Algebraic construction of the solution, capable aslo to work till the limit for $K \rightarrow \infty$ with an infinite number of digit, so it is not an approximation (that is just a technical problem as the same happens when you've as result f.ex. $\sqrt{2}$ and you've to show the value of the numerical result so someone).

Returning to our example, the Solvable Quintic: the Modulus we must use to find the Roots in the Recursive Difference from the known constant, here: -1 looking to have $Rest = 0$ (or as close to zero as we can in case of Irrational Roots) is:

$$M_{x^5-x,K,i} = -5 * \frac{x^4}{K} + 10 * \frac{x^3}{K^2} - 10 * \frac{x^2}{K^3} + 5 * \frac{x}{K^4} - \frac{2}{K^5}$$

That differs from the 5th line of the Tartaglia's triangle $M_{5,K,i}$ (represent X^5), for the last term, just, since we have to add with the right sign $M_{1,K,i}$ (represent X), that is the constant $\frac{1}{K}$ summed K times.

And still if the Number R_x* represent the Closest Rational to an irrational Solution comes from a long computation of very long Rational digit numbers, we can affirm that we can always find any R_x Root, using what I hope is clear now is a more General Root (Algebraic) Extractor (GRE theorem).

So the point is: I've produced a new more general definition for a Radical Root, finding a General Polynomial Root Extractor that can works on any Polynomial and the classic n-th Root algo that is capable to work (in general) just on Hypercube, is just a sub class of this General Polynomial Root Extractor.

Unfortunately the GRE method can works on any polynomial equation, but NOT ANY EQUATION will produce a first integer derivative that has the same zero of the original equation.

Here is the case of:

$$(x-1)(x-2)(x-3)(x-5)(x-7)=0$$

that lead to a rest of -4 instead of the first zero, as soon as you start the descent, and then no more zeros will appear:

	((x-7)(x-5)(x-3)(x-2)(x-1))		
F(X)=	**x^5-18x^4+118x^3-348x^2+457x-210**		
F'i(x)=	**5x^4-72x^3+462x^2-1122x+941**		R. Difference
			210
1		214	-4
2		49	-53
3		194	-247
4		517	-764
5		1006	-1770
6		1769	-3539
7		3034	-6573
8		5149	-11722
9		8582	-20304

Here 2 example of Quintic we can solve via this algo:

$$((x-1)(x-3)(x-5)(x-7)(x-13))$$

and is:

$$x^5 - 29x^4 + 294x^3 - 1294x^2 + 2393x - 1365$$

that has as first integer derivative (or general polynomial extractor):

$$5x^4 - 126x^3 + 1066x^2 - 3591x + 4011$$

The second comes from:

$$((x-1)(x-3)(x-5)(x-7)(x-17))$$

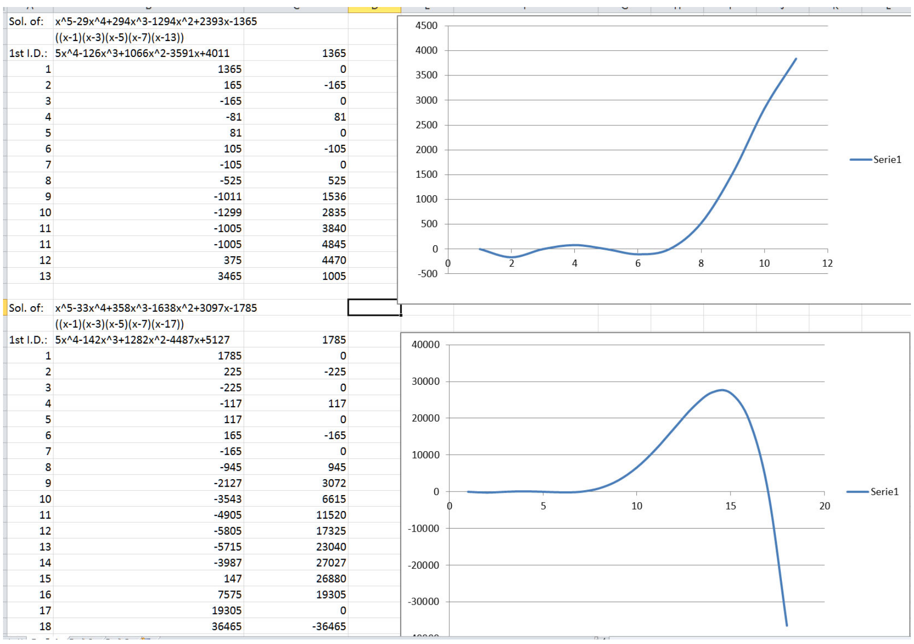
and is:

$$x^5 - 33x^4 + 358x^3 - 1638x^2 + 3097x - 1785$$

that has as first integer derivative (or general polynomial extractor):

$$5x^4 - 142x^3 + 1282x^2 - 4487x + 5127$$

Here the recursive difference show zeros:



Sol. of:	x^5-29x^4+294x^3-1294x^2+2393x-1365	
	((x-1)(x-3)(x-5)(x-7)(x-13))	
1st I.D.:	5x^4-126x^3+1066x^2-3591x+4011	1365
1	1365	0
2	165	-165
3	-165	0
4	-81	81
5	81	0
6	105	-105
7	-105	0
8	-525	525
9	-1011	1536
10	-1299	2835
11	-1005	3840
11	-1005	4845
12	375	4470
13	3465	1005

Sol. of:	x^5-33x^4+358x^3-1638x^2+3097x-1785	
	((x-1)(x-3)(x-5)(x-7)(x-17))	
1st I.D.:	5x^4-142x^3+1282x^2-4487x+5127	1785
1	1785	0
2	225	-225
3	-225	0
4	-117	117
5	117	0
6	165	-165
7	-165	0
8	-945	945
9	-2127	3072
10	-3543	6615
11	-4905	11520
12	-5805	17325
13	-5715	23040
14	-3987	27027
15	147	26880
16	7575	19305
17	19305	0
18	36465	-36465

How to obtain the 1st integer derivative in both case:

Input $\text{expand}((x-1)(x-3)(x-5)(x-7)(x-13))$

Output $x^5-29x^4+294x^3-1294x^2+2393x-1365$

Input $\text{expand}(x^5-(x-1)^5-29(x^4-(x-1)^4)+294(x^3-(x-1)^3)-1294(x^2-(x-1)^2)+2393(x-(x-1)))$

Output $5x^4-126x^3+1066x^2-3591x+4011$

Input $\text{expand}(5x^4-10x^3+10x^2-5x+1-(116x^3-174x^2+116x-29)+(882x^2-882x+294)-(4786x-2393)+(2393))$

Output $5x^4-126x^3+1066x^2-5789x+5110$

Input $\text{expand}((x-1)(x-3)(x-5)(x-7)(x-17))$

Output $x^5-33x^4+358x^3-1638x^2+3097x-1785$

Input $\text{expand}(x^5-(x-1)^5)$

Output $5x^4-10x^3+10x^2-5x+1$

Input $\text{expand}(33(x^4-(x-1)^4))$

Output $132x^3-198x^2+132x-33$

Input $\text{expand}(358(x^3-(x-1)^3))$

Output $1074x^2-1074x+358$

Input $\text{expand}(1638(x^2-(x-1)^2))$

Output $3276x-1638$

Input $\text{expand}(3097(x^1-(x-1)^1))$

Output 3097

Input $\text{expand}(5x^4-10x^3+10x^2-5x+1-(132x^3-198x^2+132x-33)+(1074x^2-1074x+358)-(3276x-1638)+(3097))$

Output $5x^4-142x^3+1282x^2-4487x+5127$

I left here also my first tricks to solve some easy n-th degree equations.
There are no big news in this method, that is quite trivial, due to the fact that is the same, or worst, in terms of computation load, to check for the solution of a polynomial, checking every possible Integer Root from 1 to the biggest possible Root, but it can be interesting to better search for the reasons let this works, and how - if, it can gives us some Rational - Algebraic Irrational Root too, once we will use the $M_{n,K}$ Rational Modulus, will also re-define what we call an Algebraic Root.

Hereafter a simple example of one of the possible solving method for a Polynomial equation of 3th degree.

B) How to find the integer Roots of 3th degree polynomial:

$$X^3-10X^2+31X-30=0$$

using the balancing method, so moving terms in both side.
Here we start with:

$$X^3=10X^2-31X+30$$

First of all remember the Sum property $X^3=\sum_1^X 3i^2-3i+1$,
and the Linearization Rule here for $n=3$ (Remember in the Vol.1 there is the general rule for all n=odd and the one one for all n=even), so we can write the Left Hand Term as:

$$X^3 = X * \sum_1^X 2i - 1$$

Than we look for the first integer root: R1=X, so we can write, remembering that $x^2 = \sum_1^x (2i-1)$:

$$R1 * \sum_{i=1}^{R1} 2i - 1 = 10 * \sum_1^{R1} (2i-1) + 30 - 31 * R1$$

so taking the squares in the same hand:

$$(10 - R1) * \sum_1^{R1} 2i - 1 = 31R1 - 30$$

So we immediately can see that the Biggest Root is bounded:
Max Root Value $(10-1) = 9$
and we have for so to solve the Complicate Modulus Equation:

$$\sum_1^{R1} 2i - 1 = (31R1 - 30)/(10 - R1)$$

Remembering we can extract the Square Root (from anywhere) using my Recursive Difference (so subtracting $2i-1$ terms from 1 to Rx)
We start the solving algo:

$$((31R1 - 30)/(10 - R1)) - (2i-1)|i = 1 =?0$$

- So we perform the 1st turn subtracting: $(2i-1)|i = 1 = 1$

$$((31R1 - 30)/(10 - R1)) - 1 =?0 \tag{a1}$$

$$((31R1 - 30 - 10 + R1)/(10 - R1))) =?0$$

$$(32R1 - 40)/(10 - R1) =?0 \tag{a2}$$

taking out the easy factors:

$$8(4R1 - 5)/(10 - R1) =?0$$

let us see that no integer solution is possible, So we go on:
-So we perform the 2nd turn subtracting: $(2i-1)|i = 2 = 3$ FROM THE (a2)

$$((32R1 - 40)/(10 - R1)) - 3 =?0$$

$$((32R1 - 40 - 30 + 3R1)/(10 - R1))) =?0$$

$$(35R1 - 70)/(10 - R1) =?0 \tag{a3}$$

taking out the easy factors:

$$35(R1-2)/(10-R1) =?0$$

Let us see that R1=2 is the 1st Integer Root.
but we can go over since we se Rx is bounded by the Max Root Value $(10-1)=9$
-So we perform the 3th turn: $(2i-1)|i=3=5$ FROM THE (a3), looking this time for the 2nd root $R2$:

$$(35R2-70)/(10-R2)-5 =?0$$

$$((35R2-70-50+5R2)/(10-R2))) =?0$$

$$(40R2-120)/(10-R2) =?0 \tag{a4}$$

taking easy out the factors:

$$40(R2-3)/(10-R2) =?0$$

Let us see that $R2=3$ is the Second Integer Root.

And we can, of course, go ahead since we see $R3$ is bounded by the Max Root Value $(10-1)=9$

So we finally find all the 3 Integer Roots, if any.

C) How to find the integer Roots of bigger degree polynomial:

To proceed we need a more powerful method (still very trivial !): for example we can use the property starts all this work:

Each Square of an integer A, is the Sum of the First "A" Odds, so we can rewrite the equation:

$$x^4 - 11x^3 + 41x^2 - 61x + 30 = 0$$

as:

$$x^2 * (x^2 - 11x + 41) = 61x - 30$$

$$x^2 = 61x - 30/(x^2 - 11x + 41)$$

Ad now start to search for the solutions remembering that each Root of the Polynomial is Equal to One of our Zeros, when we subtract 1,3,5,...(2a-1), each time one of this Number take the result of the Recursive Difference to Zero, it return us the Root of the Polynomial. So first turn is check if:

$$\frac{61x - 30}{(x^2 - 11x + 41)} - 1 =?0$$

Solving we have:

$$\frac{(61x - 30) - 1 * (x^2 - 11x + 41)}{(x^2 - 11x + 41)} =?0 \tag{1}$$

$$61x - 30 - x^2 + 11x - 41 =?0$$

$$-x^2 + 72x - 71 =?0$$

So in the classic form (where one root will not be acceptable):

$$x^2 - 72x + 71 =?0$$

$$\frac{72 \pm \sqrt[2]{72^2 - 4 * 71}}{2} = \frac{72 \pm 70}{2} = 1; (or : 71)$$

So yes $R_1 = 1$ is our first Root. Then we can subtract from the (1) the next Gnomon $(2i - 1)|_i = 2$

$$(-x^2 + 72x - 71) - 3 * (x^2 - 11x + 41) =?0 \tag{2}$$

$$-x^2 + 72x - 71 - 3x^2 + 33x - 123 =?0$$

$$-4x^2 + 105x - 194 =?0$$

That we can return in the classic form:

$$4x^2 - 105x + 194 =?0$$

$$\frac{105 \pm \sqrt[2]{105^2 - 4 * 4 * 194}}{8} = \frac{105 \pm 89}{8} = 2; (or : 24, 25....)$$

The same for the following roots...

Of course nothing change if we use the property that a Cube is the Sum of the following $M_3 = (x^3 - (x-1)^3)$ Gnomons:

$$x^3 * (x - 11x) = -41x^2 + 61x - 30$$

So we can check if holds true that:

$$x^3 = \frac{-41x^2 + 61x - 30}{(x - 11)} - 1 = ?0$$

$$-41x^2 + 61x - 30 - 1 * (x - 11) = ?0$$

$$41x^2 - 60x + 19 = ?0$$

$$\frac{60 \pm \sqrt[2]{60^2 - 4 * 41 * 19}}{82} = \frac{60 \pm 22}{82} = 1; (or : 19/41)$$

I write here an example of how to solve the case $n = 5$:

$$x^5 - 18x^4 + 118x^3 - 348x^2 + 457x - 210 = 0$$

Can be reduced to:

$$x^3 * (x^2 - 18x + 118) = 348x^2 - 457x + 210$$

so we can start from $R_1 = 1$ our search so if:

$$348x^2 - 457x + 210 - 1 * (x^2 - 18x + 118) = ?0$$

$$347x^2 - 439x + 92 = ?0$$

$$\frac{439 \pm \sqrt[2]{439^2 - 4 * 347 * 92}}{2 * 347} = \frac{439 \pm 255}{2 * 347} = 1; (or : 92/347)$$

...and of course we can go ahead searching for the next Roots.

We can go ahead ab infinitum ? Unfortunately, of course, Not, due to what is known as the Radical Closure, and the fact that we are using as final solving equation the Square, or the Cubic Solving formula.

...BUT WE CAN CHEAT and go ahead for $n > 5$!!! You can't believe ? Just remember that if $R_1 = 1$ then

$$x^6 - 24x^5 + 226x^4 - 1056x^3 + 2545x^2 - 2952X + 1260 = 0$$

is equal to write:

$$24x^5 - 226x^4 + 1056x^3 - 2545x^2 + 2952X - 1260 = 1^6$$

So:

$$24x^5 - 226x^4 + 1056x^3 - 2545x^2 + 2952X - 1261 = 0$$

we can solve as the last quintic here above:

$$x^3(24x^2 - 226x + 1056) = 2545x^2 - 2952X + 1261$$

$$2545x^2 - 2952X + 1261 - 1 * (24x^2 - 226x + 1056) =?0$$

$$2521x^2 - 2726X + 205 = 0$$

$$\frac{2726 \pm \sqrt[2]{2726^2 - 4 * 2521 * 205}}{2 * 2521} = \frac{2726 \pm 2316}{2 * 2521} = 1; (or : 205/2521)$$

etc....

Moreover, any Ring has the same properties shown here for Polinomials, will lead to similar zeros, or Roots. But this will require an Abstract Algebra discussion that is not what I would like to do here, since my purphose is to introduce young students to a new point of view or the known classic algebra.

For those will immediately argue that this is a Numerical Solution, I answer that this is an Algebraic solution, and also the extraction of a known n-th Root, i prove is a special case of this more general one, at the end of a solvable equation, will require, in non trivial case, tables or a Numerical computtion to extract the root or the n-th root.

So I hope I've proved here that the concept of Root is much wider and lead to new interesting simple results can be presented without any abstract definition of Groups / Rings / Ideals etc...

Chapt.20: The New Complicated Risen Modulus M_{n+}

To left unchanged the result of a sum:

$$P^n = \sum_1^P 3X^2 - 3X + 1$$

Lowering the Upper Limit, from P to p

Imply we Must Rise All the Terms of (for example here) M_3 of a scaling factor (as seen) P/p

Or, and this is what is interesting, to use a new one we will call M_3+, using an "unknown" method, we imagine will depends on how the Complicate Modulus M_n is build, and behave, moving backward.

In this case can be useful the develop of $((X+1)^n - X^n)$ since (in this case) it leads to the first approximation for M_{3+}:

since it is the sum of the Known modulus M_3 and of the Second derivative $Y'' = 6x$ or:

$$M_{3+} = M_3 + Y'' = M_3 + 6X = 3X^2 + 3X + 1$$

But this Linear Shift is not enough since we know the Second Integer derivative for $n = 3$ is $Y_i'' = 6X - 6$,

than the next following terms will depends on the Second Integer derivative $Y_i'' = 6X - 6$, where we have now to change $X = (x - 1)$

So the Shift (given by the variable exchange), imply we need to use also another Riser Term equal (for $n = 3$) to:

$$Y_{i,\delta}'' = 6(x-1) + 6$$

But as we saw on the Table, the Modification of All the Terms using this Two Modular Parameter, IS NOT ENOUGH to Rise To the Genuine Power, since using the new $M_{3+,\delta}$ modulus, we need to introduce a ONE TIME Correction PARAMETER we know is the Rest $= R$, that introduce the correction for the removed terms, so is equal to δ^n.

As shown in the Tables in this case $\delta^3 = 1, 8, 27, 64....$:

Here the table of what happen Lowering the Upper Limit of 1,2,3 and 4.

As we immediately see, using the Induction, will be possible to build and prove all the next case for higher δ, n.

In physics this looks like the Hysteresis Cycle: it rise following a known function, that, since we always spend (dissipate) energy, will differs once we return (felling) to the initial point. The area between the two curve represents the dissipated energy.

	M{3,δ}= {3(x)^2-3(x)+1}+ {δ^3+6δx+(3δ^2-3δ+1)}				R	
x	3x^2-3x+1	Sum	3(x)^2+3(x)+1	SUM	R=δ^3	SUM +R
1	1	1	7	7	1	8
2	7	8	19	26		27
3	19	27	37	63		64

δ=2 M3(δ=2) X=x-2

x	3x^2-3x+1	Sum	3(x)^2+9(x)+7	SUM	R	SUM+R
1	1	1	19	19	8	27
2	7	8	37	56		64
3	19	27	61	117		125
4	37	64	91	209		216

δ=3 M3(δ=3) X=x-3

x	3x^2-3x+1	Sum	3(x)^2+15(x)+19	SUM	R	SUM+R
1	1	1	37	37	27	64
2	7	8	61	98		125
3	19	27	91	189		216
4	37	64	127	316		343
5	61	125	169	485		512

δ=4 M3(δ=4) X=x-4

x	3x^2-3x+1	Sum	3(x)^2+21(x)+37	SUM	R	SUM+R
1	1	1	61	61	64	125
2	7	8	91	91		216
3	19	27	127	127		343
4	37	64	169	169		512
5	61	125	217	217		729
6	91	216	271	271		1000

δ=5 M3(δ=4) X=x-5

x	3x^2-3x+1	Sum	3(x)^2+27(x)+61	SUM	R	SUM+R
1	1	1	91	91	125	216
2	7	8	127	127		343
3	19	27	169	169		512
4	37	64	217	217		729
5	61	125	271	271		1000
6	91	216	331	331		1331

Of course, since this is an approximated formula, one can also use other formulas leading for example to having other Rest, for example reducing it as much as possible (as I'll show in a while going Rational).

I skip the proof by induction because, once again, all depends on the Binomial Develop Rule.

And still if will be interesting to see the parallel to what was known as the "Infinite Descent" since I've also shown in the Vol.1 how to go Rational with the Sum, one can imagine and try to rewrite all in terms of my Rational Step Sum, seeing that moving of $1/K$ step, rising K we can reduce the $Rest$ to Littlest and Littlest values, till 0 when we push at the limit for $K \to \infty$ and we will no longer have a REST.

Since I hate professors leave to the Diligent Students the Rest of the Proof (as an exercise) I'll do all the (trivial) job as soon as I have time.

Chapt.21: List of Known, and New Rules for Sums and Step Sums

Here I remember a short list of the known Sum's rules. Some of them will LEFT UNCHANGED the RESULT of the NEW SUM, some will modify it. All this are process that Cut, Split, Stretch, Scale the Area Bellow the first derivative so we can figure them out also painting a picture on a Cartesian Plane.

1 - Ordinal Rules, based on known properties:

Rule 1.1 Calling M_n the Integer Complicate Modulus (that will be the Ordinal Number for Power of Integers):

$M_n = (x^n - (x-1)^n)$

And Calling $M_{n,K}$ the Rational Complicate Modulus (Ordinal for Power of Rational):

$$M_{n,K} = \binom{n}{1}\frac{x^{n-1}}{K^m} + \binom{n}{2}\frac{x^{n-2}}{K^{2m}} + \binom{n}{3}\frac{x^{n-3}}{K^{3m}} + ... + / - \frac{1}{K^{n*m}}$$

if $A \in \mathbb{N}^+$ then:

$$A^n = \sum_{x=1}^{A} M_n = \sum_{x=1/K}^{A} M_{n,K} = \lim_{K\to\infty} \sum_{x=1/K}^{A} M_{n,K} = \int_{x=0}^{A} nx^{n-1}dx$$

Rule 1.2: if $A = \left(\frac{P}{K}\right)$ with $P, Q \in \mathbb{N}^+$ so if $A \in \mathbb{Q}^+ - \mathbb{N}^+$ then we can write A^n as

$$A^n = \left(\frac{P}{K}\right) = \sum_{x=1/K^m}^{A} M_{n,K^m} = \lim_{K\to\infty} \sum_{x=1/K}^{A} M_{n,K^m} = \int_{x=0}^{A} nx^{n-1}dx$$

Rule 1.3: if $A \in \mathbb{R} - \mathbb{Q}^+$ and $A = KnownIrrational$ then we can again use the Rule 1.2, having an irrational Step Sum, so having Irrational Lower and Upper Limit, and a Finite Integer Number $A * K^m$ of Irrational Step.

$$A^n = \sum_{x=1/K}^{A} M_{n,K} = \lim_{K\to\infty} \sum_{x=1/K}^{A} M_{n,K} = \int_{x=0}^{A} nx^{n-1}dx$$

Rule 1.4: if $A \in \mathbb{R} - \mathbb{Q}^+$ and $A \neq KnownIrrational$ then we have just one way to write it, so via Limits so Integrals:

$$A^n = \lim_{K\to\infty} \sum_{x=1/K}^{A} M_{n,K} = \int_{x=0}^{A} nx^{n-1}dx$$

2 - Interesting, known, Rules (In Number Theory there are, of course, more known rules).

One of those we need to remember is: **an integer A has an unique factorization**

Rule 2.1: If A is not a prime, so for example $A = \pi_1 * \pi_2$ than its power can be represented as a product of proper Sums.
Taking as example A^2:

$$A^2 = \sum_{x=1}^{A} (2x - 1)$$

Follows immediately that it can be divided in a product of 2 sums

$$A^2 = (\pi_1)^2 * (\pi_2)^2 = \sum_{x=1}^{\pi_1} (2x - 1) * \sum_{x=1}^{\pi_2} (2x - 1)$$

Of course more factors, more sums.

2.2: If $A^n = \pi_1^n + \pi_2^n$

then also the sum can be divided, but this happen just for $n = 2$ as a consequence of Fermat's Last Theorem Proof:

$$A^2 = \sum_{x=1}^{A} (2x - 1) = \sum_{x=1}^{\pi_1} (2x - 1) + \sum_{x=1}^{\pi_2} (2x - 1)$$

Rule 3: How to GROUP, or CUT, a SUM :

Sums are interesting since is very easy to be Grouped or Cut without changing the result:

Starting from the known in case $n = 2$ and $A = \pi_1 * \pi_2$ we can re-use the sum identity we already know from the RULE 2.1:

$$A^2 = \sum_{x=1}^{\pi_1} (2x - 1) * \sum_{x=1}^{\pi_2} (2x - 1) =$$

And since independently by how we call the index x or r, it is a mute variable, we can use just one index: x, to have:

$$A^2 = \sum_{x=1}^{\pi_1} (2x - 1) * \sum_{x=1}^{\pi_2} (2x - 1) =$$

$$= \sum_{x=1}^{\pi_1} (2x - 1) * \left(\sum_{x=1}^{\pi_1} (2x - 1) + \sum_{x=\pi_1+1}^{\pi_2} (2x - 1) \right)$$

I remember now how is possible to manipulate a Sum equal to a Power of A without changing the result of the Sum, so having back again the same Power of Integer, or Rational A.

$$\pi_1^2 * \left(\sum_{x=1}^{\pi_1} (2x-1) + \sum_{x=\pi_1+1}^{\pi_2} (2x-1) \right) = (\pi_1)^2 * (\pi_2)^2$$

so:

$$\sum_{x=1}^{\pi_1} (2x-1) + \sum_{x=\pi_1+1}^{\pi_2} (2x-1) = (\pi_2)^2$$

That's very easy but will help us in the next tricks I'll present.

Rule.4: Multiply the Sum by A is equal to Multiply All the internal Terms by A (so both type index dependent and constant ones)

$$\left(\sum_{x=1}^{A} (2x-1) \right) * A = \sum_{x=1}^{A} (2xA - A)$$

Rule.5: If we Change ONLY the upper limit of the Sum:

This manipulation will produce different effects on the result of the sum:

- Rule 5.1 If we multiply the Upper Limit A by itself, or if we make a Power of it :

If n is the exponent of the original sum:

$$A^n = \sum_{x=1}^{A} (x^n - (x-1)^n)$$

If we change the Upper Limit from A to A^p we have:

$$\sum_{x=1}^{A^p} (x^n - (x-1)^n) = A^{(pn)}$$

Example 1: if we have:

$$A^2 = \sum_{x=1}^{A} (2x - 1)$$

then taking as new upper limit A^2 we have:

$$\sum_{x=1}^{A*A} (2x - 1) = A^4$$

- Rule 5.2 What happen if we multiply the upper limit A by an integer P :
Is trivial again: the result of the sum change from A^n to $(A * P)^n$

Rule.5.3: Multiply Both the Lower and the Upper Limit by an integer P change the Result of the Sum.

For example if we keep:

$$A^2 = \sum_{x=1}^{A} (2x - 1)$$

if we multiply both lower and upper limit by P:

$$\sum_{x=1*P}^{A*P} (2 * P * X - 1)$$

We have no longer a square:

Table 15: Add caption

x	**X = 5 x**	**2X-1**	**SUM**	**new Square: P*x**
1	5	9	9	3
2	10	19	28	5,291503
3	15	29	57	7,549834
4	20	39	96	9,797959
5	25	49	145	12,04159
6	30	59	204	14,28286
7	35	69	273	16,52271
8	40	79	352	18,76166
9	45	89	441	21
10	50	99	540	23,2379

As you can see sometimes we have a Square again, but I left the interesting concerning on what happen to the Vol.2

To Let the Sum give back again the same Power we just need to make the exchange of variable $X = Px$, so divide each x dependent term by P (at the same power).
For example if we keep:

$$A^2 = \sum_{x=1}^{A} (2x - 1)$$

The same Power with Shifted, multiplied by P, Limits will be:

$$A^2 = \sum_{X=1*P}^{A*P} (2 * \frac{X}{P} - 1)$$

Table 16: Add caption

x	$X = 5x$	$2*X/5-1$	SUM
1	5	1	1
2	10	3	4
3	15	5	9
4	20	7	16
5	25	9	25
6	30	11	36
7	35	13	49
8	40	15	64
9	45	17	81
10	50	19	100

Or if we wanna have back a Power that is P times bigger, we need to make the exchange of variable X=Px, plus the Right Shift into the constant term in this way:

Starting from:
For example if we keep:

$$A^2 = \sum_{x=1}^{A} (2x - 1)$$

- **Rule 5.2** if we multiply both lower and upper limit by P, and we wanna be sure we will have always back a Power of an integer we need to arrange the Constant Term too as in the following example:

$$\sum_{x=1*P}^{A*P} (2 * P * X - P^2) = (A * P)^2$$

The following RULES, valid for the SUM that are EQUAL to a POWER of INTEGERS, are probably less known, and will be useful for solving Several (also very Hard) Number Theory Problems, as we will see in Vol.2: RATIONAL ANALYSIS.

It's possible to arrange, under certain conditions, the Sum Limits, and the internal Terms of the Sum to Left Unchanged the result of the Sum, Just if WE RESPECT some RULES:

Rule.6: Shifting of a fix value A, Both the Lower and the Upper Limit: is equal to ADD, (or subtract) A to the Index dependent terms X where present.

- 6.1 If the index in the sum is x, Rising both Lower and Upper Limit of A is equal to change X in $(X - A)$.

Here an example for $n = 3$ on how to do if we Rise Both the limits by A and we have to leave unchanged the result of the sum:

$$B^3 = \sum_{X=1}^{B} 3X^2 - 3X + 1 = \sum_{X=A+1}^{A+B} [3(X - A)^2 - 3(X - A) + 1]$$

The proof is simple: taking this example as reference the shift doesn't affect the number of step, that rest the same:

$$(B - 1) = A + B - (A + 1)$$

Table 17: Add caption

	X	$3X^2 - 3X + 1$	**SUM**	$3(X - A)^2 - 3(X - A) + 1$	**SUM**
	1	1	1		
	2	7	8		
	3	19	27		
	4	37	64		
A	5	61	125		
A+1	6	91	216	1	1
	7	127	343	7	8
	8	169	512	19	27
	9	217	729	37	64
	10	271	1000	61	125
	11	331	1331	91	216
	12	397	1728	127	343
	13	469	2197	169	512
	14	547	2744	217	729
	15	631	3375	271	1000
	16	721	4096	331	1331
B	17	817	4913	397	1728
	18	919	5832	469	2197
	19	1027	6859	547	2744
	20	1141	8000	631	3375
	21	1261	9261	721	4096
A+B	22	1387	10648	817	4913

- 6.2 If the index in the sum is x, Reducing both Lower and Upper Limit of B is equal to change x in $(x + B)$. So if the term is $3X^2$, the new term will be $3(X + B)^2$

Here an example for n=3 on how to do if we LOWER BOTH the limits by B:

$$\sum_{X=B+1}^{C} 3X^2 - 3X + 1 = \sum_{x=1}^{C-B} [3(X + B)^2 - 3(X + B) + 1]$$

The proof is simple: taking this example as reference the shift doesn't affect the number of step, that rest the same:

$$C - (B + 1) = C - B - 1$$

And the shift affect just each term of the sum that is "index dependent" so instead of x we simply put $x + B$ and nothing change.

As told we put $(x - B)$ in case we want to RISE both the Limits of B

Rule.7: Any n-th power of integer is equal to a Sum of a linear terms; Odds or Even powers require different linear terms:

Rule.7a: Any EVEN n-th power of integer $A^{(2P)}$ is equal to a Sum of a linear terms $(2x - 1)$

Rule.7b: Any ODD n-th power of integer $A^{(2P+1)}$ is equal to a Sum of a linear terms $(2xA - A)$

ALL THE PREVIOUS RULES can be now extended to the integral / derivative process I show pushing Sum's to the limit to discover that all that rules are already well known, since the infinitesimal calculus has proceeded faster than this "trivial" Rational play.

Rule 8: The Sum can be transformed in a Step Sum , Step $1/K$:

In case $A = \frac{P}{K}$, we can write:

$$A^n = \left(\frac{P}{K}\right)^n = \sum_{x=1/K}^{P/K} M_{n,K}$$

where: where:

$$M_{n,K} = \binom{n}{1}\frac{x^{n-1}}{K^m} - \binom{n}{2}\frac{x^{n-2}}{K^{2m}} + \binom{n}{3}\frac{x^{n-3}}{K^{3m}} + ... + / - \frac{1}{K^{n*m}}$$

Rule 9.1: The Step Sum, step $1/K$, can be transformed in an Riemann (like) Integral just passing to the limit for $K \to \infty$:

$$A^n = \lim_{k\to\infty} \sum_{x=1/K}^{A} M_{n,K} = \int_0^A (n * x^{n-1})dx$$

Rule 9.2: As the the Recursive Sum becomes an Integral at the limit for $K \to \infty$, the Recursive Difference becomes at the limit for $K \to \infty$ the derivative.

Rule 10 as extension of Rule 6: Shifting of a fix value (B for example), both the lower and the upper limit is equal to ADD B to the Index (here x). So the Shift affects only in the Index dependent Terms . Here an example for n=3 where we know that:

$$\sum_{X=B+1}^{C} 3X^2 - 3X + 1 = \sum_{X=1}^{C-B} [3(X+B)^2 - 3(X+B) + 1]$$

We can see now that this rule works also passing to the Step Sum, Step $1/K$, so putting $x = X/K$ and then to the integral that is the limit of the Step Sum for $K \to \infty$:

Proof:

Starting from:

$$\lim_{K\to\infty} \sum_{x=B+1/K}^{C} \left(\frac{3x^2}{K} - \frac{3x}{K^2} + \frac{1}{K^3} \right) =$$

$$= \int_B^C (3x^2)dx = x^2|(C,B) = C^3 - B^3$$

Shifting the lower and the upper limit by B, and adding B at the index (here x) dependent terms (only) we have again:

$$\lim_{k\to\infty} \sum_{x=1/k}^{C-B} [3(x+B)^2/k - 3(x+B)/k^2 + 1/k^3] =$$

$$= \int_0^{C-B} [3(x+B)^2]dx = C^3 - B^3$$

And in general for the infinitesimal Step dx, so in case we push the Sum to the Integral we can write :

$$\int_B^C [n * x^{(n-1)}]dx = \int_0^{C-B} [n(x+B)^{(n-1)}]dx = C^n - B^n$$

While of course for the special case $n = 2$ for some triplets known as Pythagorean Triplets the relation holds true also for Sums having Integers Step:

$$\sum_{X=B+1}^{C} 2X - 1 = \sum_{X=1}^{C-B} [2(X+B) - 1]$$

Is clearly true for any Pythagorean Triplet, f.e.x $A = 3$, $B = 4$, $C = 5$.

And this because both Terms ad Limits Linearly behave.

Author Note: This can be, probably, the evidence that Fermat, while studying the properties of Powers and Integrals, got himself to this conclusion.

Of course at that time justifying the "vanishing" terms (since major orders infinitesimal quantity) to his colleagues was an impossible mission, so probably this can be the reason why we haven't found his concerning about.

In the previous Rules we have seen how to Shift the Lower Limits Leaving unchanged the result. Here an example for n=3:

$$\sum_{B+1}^{C} 3x^2 - 3x + 1 = \sum_{1}^{C-B} 3(x+B)^2 - 3(x+B) + 1$$

But while we are sure that the equality work, we do not ask ourself to what this value correspond, so for example if it can be (again) equal to a Cube, or not. And this exactly what is known as Fermat The last Theorem:

$$A^3 = \sum_{1}^{A} 3x^2 - 3x + 1 =? \sum_{B+1}^{C} 3x^2 - 3x + 1 = \sum_{1}^{C-B} 3(x+B)^2 - 3(x+B) + 1$$

Where we are sure that the equality of the first two terms holds, and the same for the equality of the last two terms, but NOT of the equality between the first two, with the last two. Fermat is for so a Special Case of a most general Shifting Rule. Wiles prove the equality is impossible and I'll prove impossible too, for all $n > 2$ in a most simple way in the Vol.2, after presenting here this last Rule.

Rule 11: Scaling the Sum. Index Vs. Terms Scaling / Shifting Rules

We see now the Last Set of Rules will help us to work with any problem involves Powers and Equalities:

A) - how to Scale (Up or down) the Upper Limit LEAVING THE RESULT UNCHANGED, so Rising/Lowering the Internal Terms of the SUM (JUST).

And, what happen trying to apply two modifications so:

B) - how to Scale (Up or down) the Upper Limit AND shifting the Lower one, LEAVING THE RESULT UNCHANGED, so Rising/Lowering the Internal Terms of the SUM (JUST), that is what Fermat state in his equation.

So in other terms for the Scaling Rule A:

A1) Is it possible, and under which conditions, to: Lower the UPPER LIMIT from A to $a < A$, just, leaving the result unchanged RISING the VALUE of the INTERNAL TERM/s ?

A2) Or, vice versa, is it possible, and under which conditions, to: Rise the LOWER LIMIT, for example from 1 to $LL > 1$, just LOWERING the VALUE of the INTERNAL TERM/s ?

The answer, for both case, is of course YES, with a trivial solution, if we introduce the Lowering/Rising Factor $\rho = (A/a)$:

$$\sum_{1}^{A} M_n = \sum_{1}^{a=A/\rho} \left(\frac{A}{a}\right)^n M_n = \sum_{1}^{a} \rho^n M_n = \rho^n \sum_{1}^{a} M_n$$

$$\sum_{1}^{a} M_n = \sum_{1}^{A=a*\rho} \left(\frac{a}{A}\right)^n M_n = \sum_{1}^{A} (1/\rho)^n M_n = (1/\rho)^n \sum_{1}^{a} M_n$$

As we can see the Lowering Factor $\rho = (A/a)$ is of the same degree of the n-th Power we are working on, and is applied on all the terms of the Sum. The Factor can be, clearly, taken out from the Sum using the well known Sum's Rule.

Special Case if $\rho = n$

Will also be immediately clear that if $\rho = n$ it is also a factor of the binomial develop so, for example:

$$\sum_1^a M_n = \sum_1^A (A^n/a^n) M_n = \sum_1^A (1/\rho) M_n = (1/\rho) \sum_1^a M_n$$

With $a = 3$, $A = 9$, $\rho = (A^3/a^3 = 27/9 = 3 = n)$ can be written as::

$$a^3 = \sum_1^a 3x^2 - 3x + 1 = A^3\rho = \sum_1^A (3/\rho)x^2 - (3/\rho)x + 1/\rho) = \sum_1^9 x^2 - x + 1/3$$

This reduction can be done each time $\rho = (A^n/a^n) = n = prime$ due to the the Binomial Develop property that for all $n = Prime$, all the binomial develop terms (different from 1) has n as common factor.

Table 18: Introducing the ρ factor in the terms

X	$3X^2 - 3X + 1$	**SUM**	$X^2 - X + 1/3$	**SUM**
1	1	1	0,333333333	0,333333
2	7	8	2,333333333	2,666667
3	19	27	6,333333333	9

Let N be prime, we can prove that:

$\binom{N}{k}$ is divisible by N for $k = 1, 2, \ldots, (N-1)$

Let $M = \binom{N}{k}$ then

$$M = \frac{N!}{k!(N-k)!}, \quad \text{or equivalently} \quad N! = MK!(N-k)!$$

Clearly N divides $N!$

Thus N divides $M * k!(N-k)!$.

But if a Prime divides a product, then it divides at least one of the terms. Since N cannot divide $k!$ or $(N-k)!$, it must divide M.

But is there any way to re-write this formula in another way, without introducing the "trivial" ρ factor, for example changing the Index Dependent Terms Only ?

The answer is, in general, OF COURSE NOT, since, for example, once we apply the correction on the constant term, or on the Index Dependent Terms, only, the correction is, for sure, not a General Solution, because rising x of 1, or else, will immediately change the result of the equation, so it is no longer a general formula, but, in case, a special solution.

So the target is to find an Approximated Formula and, better, the Most Approximated Formula that fixed the problem under certain conditions and that works for ALL the x of the same problem.

So the first thing we have to do is check if introducing a little as possible "Rest" we are able to use a Reasonable Right Approximated Formula, and possible the Most Approximated one that is the one (or the set of the formulas) that once pushed to the limit will perfectly fit the equation without Rest, as I did for the Classic Rational Sum.

From the Interesting Identity:

$$\sum_{x=1/A}^{A}\left(\frac{2x}{A}-\frac{1}{A^2}\right)=\sum_{x=1/A}^{B-A}\left(\left(\frac{2x}{A}*\frac{A^2}{(B-A)^2}\right)-\frac{1}{A^2}\frac{A^2}{(B-A)^2}\right)$$

Prove Pell's Equation $B^2-2A^2=1$ is:

$$\sum_{x=1/A}^{A}\left(\frac{2x}{A}-\frac{1}{A^2}\right)=\sum_{x=1/A}^{B-A}\left(\left(\frac{2x}{A}+1\right)-\frac{1}{A^2}\right)$$

Has a Minimal Solution, then infinite solutions with $A, B \in \mathbb{N}$

How to find a non trivial solution to Pell's Equation $B^2 - 2A^2 = 1$

In Sum:

$$\sum_{X=1}^{B}(2X-1) - 2 * \sum_{X=1}^{A}(2X-1) = 1$$

$$\sum_{X=1}^{B}(2X-1) - \sum_{X=1}^{A}(2X-1) = \sum_{X=1}^{A}(2X-1) + 1$$

$$\sum_{X=A+1}^{B}(2X-1) = \sum_{X=1}^{A}(2X-1) + 1$$

Shifting the Lower limit:

$$\sum_{X=1}^{B-A}(2(X+A)-1) = \sum_{X=1}^{A}(2X-1) + 1$$

Taking out the genuine square:

$$\sum_{X=1}^{B-A}(2X-1) + 2A * (B-A) = \sum_{X=1}^{A}(2X-1) + 1$$

We are looking for a solution so first concerning we can make is what about $B - A = 1$?

$$1 + 2A = A^2 + 1$$

From where: $A = 2$ and then $B = 3$

From the point of view of my Complicate Modulus Algebra the question (under FLT conditions so: $A < B < C \in \mathbb{N}^+$):

$$A^3 =?C^3 - B^3$$

Has an immediate answer (after one understood my CMA and the proof into Vol.2): NO !

Because that will lead to the clearly false equality (in the integers)

$$\sum_{x=1/\sqrt{A}}^{A\sqrt{A}} \left(\frac{2x}{\sqrt{A}} - \frac{1}{A}\right) = \sum_{x=1/\sqrt{C}}^{C\sqrt{C}} \left(\frac{2x}{\sqrt{C}} - \frac{1}{C}\right) - \sum_{x=1/\sqrt{B}}^{B\sqrt{B}} \left(\frac{2x}{\sqrt{B}} - \frac{1}{B}\right)$$

Since there is no common factor for C and B, therefore the only common divisor for the 3 Sum (let the step rise all the 3 irrational, coprime, Upper Limits) is $1/K$ with $(K \rightarrow \infty)$ so with the known integrand factor $1/K = dx$ where it can satisfy (at the condition that one of the 3 parameter $\in \mathbb{R} - \mathbb{Q}$,

So the only way to obtain an equality is to perform the integral:

$$\int_{x=0}^{A\sqrt{A}} 2x dx = \int_{x=0}^{C\sqrt{C}} 2x dx - \int_{x=0}^{B\sqrt{B}} 2x dx$$

(c) Stefano Maruelli

Fermat the Last with Complicate Modulus Algebra:

Let $A, B, C, n \in \mathbb{N}^+$.
Fermat state that from $n = 3$ it is true that

$$C^n \neq A^n + B^n$$

We can start to observe what happens in a genuine Power Develop, where it's always possible to find A and B for what:

$$C^3 = (A + B)^3 \tag{2}$$

Because we can write:

$$\sum_{X=1}^{C}(3X^2 - 3X + 1) = \sum_{X=1}^{A+B}(3X^2 - 3X + 1)$$

Is an equality for $C = A + B$, means that it has NO LAST ELEMENT, in fact we can dismount both Sum, step by step (from both limits so in both directions) till having back $0 = 0$, so it is also true that:

$$\sum_{X=1}^{C-1}(3X^2 - 3X + 1) = \sum_{X=1}^{A+B-1}(3X^2 - 3X + 1)$$
$$\sum_{X=1}^{C-2}(3X^2 - 3X + 1) = \sum_{X=1}^{A+B-2}(3X^2 - 3X + 1)$$
$$\sum_{X=1}^{C-3}(3X^2 - 3X + 1) = \sum_{X=1}^{A+B-3}(3X^2 - 3X + 1)$$

.....

$$\sum_{X=1}^{1}(3X^2 - 3X + 1) = \sum_{X=1}^{1}(3X^2 - 3X + 1)$$

and finally:

$$0 = 0$$

Or vice versa from the Lower to the Upper Limit. What Fermat State is that if the equality it's True, than there must be another way to write A^n in terms of $C^n - B^n$ and to better understand who this terms are, we can transform them in Sum, and then we can apply on the all the Sum Rule we know now (also into Rationals) to investigate why the case $n = 2$ works, and why not, from $n = 3$. I start hereafter to show in detail the case $n = 2$ leaving all the final proof to the Vol.2 where I will show how to apply all the rules we learn here into Vol.1.

Fermat the Last $n = 2$ with Complicate Modulus Algebra CMA):

1) Why Fermat $n = 2$ admit solutions:

Let $A < B < C \in \mathbb{N}^+$, $n = 2$ Prove with the CMA that the following Fermat's equation can be true into the integers.

$$A^2 = C^2 + B^2 \tag{1}$$

If and only if $A, B, C \in \mathbb{N}^+$, then we can rewrite the (1) as,

$$\sum_{X=1}^{A}(2X-1) = \sum_{X=1}^{C}(2X-1) - \sum_{X=1}^{B}(2X-1) \tag{1b}$$

On what we can apply the Direct Cut:

$$\sum_{X=1}^{A}(2X-1) = \sum_{X=B+1}^{C}(2X-1) \tag{1c}$$

On what we can apply the Limit Shift, then expel the rest, an then again):

$$\sum_{X=1}^{A}(2X-1) = \sum_{X=1}^{C-B}(2(X+B)-1) \tag{1d}$$

$$\sum_{X=1}^{A}(2X-1) = \sum_{X=1}^{C-B}(2X-1) + 2B(C-B) \tag{1e}$$

$$\sum_{X=C-B+1}^{A}(2X-1) = 2B(C-B) \tag{1f}$$

$$\sum_{X=1}^{A+B-C}(2(X+C-B)-1) = 2B(C-B) \tag{1g}$$

$$\sum_{X=1}^{A+B-C}(2X-1) + 2(A+B-C)(C-B) = 2B(C-B) \tag{1i}$$

$$\sum_{X=1}^{A+B-C}(2X-1) = 2B(C-B) - 2(A+B-C)(C-B) \tag{1l}$$

$$\sum_{X=1}^{A+B-C} (2X-1) = 2BC - 2B^2 + 2AB - 2AC + 2B^2 - 4BC + 2C^2 \tag{1m}$$

$$A^2 + 2AB - 2AC + B^2 - 2BC + C^2 = 2BC - 2B^2 + 2AB - 2AC + 2B^2 - 4BC + 2C^2 \tag{1n}$$

$$A^2 + \cancel{2AB} - \cancel{2AC} + B^2 - \cancel{2BC} - C^2 = \cancel{2BC} - \cancel{2B^2} + \cancel{2AB} - \cancel{2AC} + \cancel{2B^2} - \cancel{4BC} + \cancel{2C^2} \tag{1o}$$

$$A^2 + B^2 - C^2 = 0 \tag{1p}$$

2) Why for the same reasons Fermat from $n = 3$ doesen't works:

Let new positive integers $A < B < C \in \mathbb{N}^+$, $n = 3$, the same as we did for $n = 2$:

$$\sum_{X=1}^{A}(3X^2 - 3X + 1) \neq \sum_{X=1}^{C}(3X^2 - 3X + 1) - \sum_{X=1}^{B}(3X^2 - 3X + 1) \tag{2}$$

in the case $n = 3$, and most in general for any $n > 2$

Because if we make the hypo (already with the first known cut):

$$\sum_{X=1}^{A}(3X^2 - 3X + 1) = \sum_{X=B+1}^{C} (3X^2 - 3X + 1) \tag{2a}$$

Starting the dismounting process for both side (that has to follow all the Sum Rules I've shown into Vol.1), we will see it stops at a last Cube, bigger than zero, proving there is an Irreducible Rest, so proving that the equation (3) is not an equality into the integers. The dismounting process it's easy, just little long tedious as following (after the direct cut):

Shift the lower limit:

$$\sum_{X=1}^{A}(3X^2 - 3X + 1) = \sum_{X=1}^{C-B}(3(X+B)^2 - 3(X+B) + 1) \tag{2b}$$

Taking out the genuine Cube of $(C - B)$, isolating the Rest:

$$\sum_{X=1}^{A}(3X^2 - 3X + 1) = \sum_{X=1}^{C-B}(3X^2 - 3X + 1) + 3B\sum_{X=1}^{C-B}(2X-1) + 3B^2(C-B) \tag{2c}$$

$$\sum_{X=C-B+1}^{A} (3X^2 - 3X + 1) = 3B\sum_{X=1}^{C-B}(2X-1) + 3B^2(C-B) \tag{2d}$$

On what we have again to shift the Lower Limit of the first Sum to 1 having (we work just onto the first sum now):

$$\sum_{1}^{A+B-C}(3(X+C-B)^2-3(X+C-B)+1)=\sum_{1}^{A+B-C}(3X^2-3X+1)+3B\sum_{1}^{A+B-C}(2X-1)-$$
$$-3\sum_{1}^{A+B-C}(2X-1)+3*(C-B)^2$$

putting the new terms again into the upper $2d$ formula, that has no space for the numeric tag, we have:

$$3B\sum_{X=1}^{C-B}(2X-1)+3B^2\sum_{X=1}^{C-B}1=?\sum_{1}^{A+B-C}(3X^2-3X+1)+3B\sum_{1}^{A+B-C}(2X-1)-3C\sum_{1}^{A+B-C}(2X-1)+3*(C-B)^2 \quad (2e)$$

Reorganizing with the cube at in the left and all the rest on the right, with all squares written as Sum:

$$\sum_{1}^{A+B-C}(3X^2-3X+1)=3B\sum_{X=1}^{C-B}(2X-1)+3(C-B)\sum_{X=1}^{B}(2X-1)- \quad (2fA)$$

$$-3B\sum_{1}^{A+B-C}(2X-1)+3C\sum_{1}^{A+B-C}(2X-1)-3*\sum_{1}^{C-B}(2X-1) \quad (2fB)$$

$$(A+B-C)^3=3(B-1)(C-B)^2+3B^2(C-B)-3B(A+B-C)^2+3C(A+B-C)^2 \quad (2g)$$

$$(A+B-C)^3=3(B-1)(C-B)^2+3B^2(C-B)-3(A+B-C)^2(C-B) \quad (2h)$$

That prove we rise a minimal, non longer reducible, Cube, but what is equal too is not what we already prove it's true:

$$A^3+3AB^2-6ABC+3AC^2+B^3+3BA^2+3BC^2-C^3-3CA^2-3CB^2=$$

$$=6AB^2-12ABC+6AC^2+3B^3-3B^2+3BA^2+12BC^2+6BC-3C^3-3C^2-3CA^2-12CB^2$$

So we prove that in case the (2) is true, we will have 2 ways to write the Binomial Develope for $(A+B-C)^3$

If you understood how CMA works, the proof is closed.

Why if you rest with the old mind, you can be convinced that we just reduce the investigation in if $(A+B-C)$ is a factor of the first two terms of the right hand (too):

$$\frac{3(B-1)(C-B)^2+3B^2(C-B)}{(A+B-C)}\in\mathbb{N}^+ \quad (2i)$$

$$\frac{(-6B^3+3B^2+3BC+3CB^2-3C)}{A+B-C} \quad (2l)$$

And one now argue that still if we will be able to prove it is not, so we (in case) prove the case $n = 3$ we have to make the same long work for any following bigger n....

I hope it is clear why we can stop to the (2H): any Genuine equation lead to an equality has 0 in both side as last element in the dismounting process, while here we have not since we have a different number of combination (Binomial and Multinomial developes are combinatoric dependent theorem...)

But how remove in you any doubt of these ? Continuing from the (2h) we have:

$$A^3 + 3AB^2 - 6ABC + 3AC^2 + B^3 + 3BA^2 + 3BC^2 - C^3 - 3CA^2 - 3CB^2 =$$
$$= 6CB^2 + 3CA^2 - 3C^2 + 3C^3 + 6BC - 6BC^2 - 3BA^2 - 3B^2 - 3B^3 - 6AC^2 + 12ABC - 6AB^2$$

So remembering we can use $C^3 - B^3 = A^3$:

$$A^3+9AB^2-18ABC+9AC^2+4B^3+3B^2+6BA^2+9BC^2-6BC-4C^3+3C^2-6CA^2-9CB^2 = 0 \tag{2m}$$

$$3A^3 = 9AB^2 - 18ABC + 9AC^2 + 3B^2 + 6BA^2 + 9BC^2 - 6BC + 3C^2 - 6CA^2 - 9CB^2 \tag{2n}$$

$$A^3 = 3AB^2 - 6ABC + 3AC^2 + B^2 + 2BA^2 + 3BC^2 - 2BC + C^2 - 2CA^2 - 3CB^2 \tag{2o}$$

$$A^3 + 2A^2(C - B) - 3A(C - B)^2 = B^2 + 3BC^2 - 2BC + C^2 - 3CB^2 \tag{2p}$$

$$A^3 + 2A^2(C - B) - 3A(C - B)^2 = (C - B)^2 + 3BC(C - B) \tag{2q}$$

that is far from the original equation:

$$A^3 + B^3 - C^3 = 0$$

we can enter into the (2q) to vanish the term A^3 having:

$$9AB^2-18ABC+9AC^2+3B^2+6BA^2+9BC^2-6BC+3C^2-6CA^2-9CB^2 = 3C^3-3B^3 \tag{2r}$$

$$3AB^2 - 6ABC + 3AC^2 + B^2 + 2BA^2 + 3BC^2 - 2BC + C^2 - 2CA^2 - 3CB^2 = C^3 - B^3 \tag{2s}$$

or:

$$A^3 = 3AB^2 - 6ABC + 3AC^2 + B^2 + 2BA^2 + 3BC^2 - 2BC + C^2 - 2CA^2 - 3CB^2 \tag{2t}$$

on where trying to rewrite all term in Sum, again, we can go on -ab infinitum- trying, but not arriving, to the condition $0 = 0$ as shown in page 180.

$$\sum_{x=1/A}^{A}(2x/A-1/A^2)=\sum_{x=1/A}^{C-B}(2(x+B)/A-1/A^2))$$

Still if it's again a simple question, we need to go deep inside the Sum Behavior to show how Index Versus Terms behave.

As in my graphic style, we start to see a numerical examples for $n=3$ of what happen in case we try to modify both the Index and the Internal Terms TRYING to Left unchanged the Result.

- We start to Lower the Upper Limit from A to $a=A-1$, and we call $\delta=A-a$

Trying to Left Unchanged the Result, we take the case $n=3$ ($n=2$ is a special case we will see later), as example, knowing that in this case the second derivative is a monotone rising curve and is Linear and equal to $Y''=6X$,

- and that the Second Integer derivative is (*) : $Y_i''=6X-6$,

(*) but remembering that from what seen in the Vol1. Chapter.12 this true just for $x>n$ and is NOT true for the First $n-1$ Terms of the SUM.

This imply that is NO LONGER (in general) possible to "REBUILD" a Genuine Power without the introduction of a CORRECTION PARAMETER, we know is a REST and we are able (with modular math) to play with.

So till now we saw $Rest=0$, just, operations, while this times we have to use our Complicate Modulus Algebra in all it's Power, so we have to be prepared to introduce/play also with a CORRECTION PARAMETER R let us re-write our formulas using the MOST APPROXIMATED MODULUS (in case a Zero cannot be found with any possible Integer/Rational formula), so using the most similar approximation formula we already know, and works, in the classic case.

So as we can see in the first Table: keeping for example the Cubes, so a Sums having as terms: $M_3=3x^2-3x+1$

- Lowering for example the Upper Limit from $A=2$ to $a=1$, so $\delta=A-a=1$

Fermat the Last with Complicate Modulus Algebra:

Let $A < B < C \in \mathbb{N}^{+}$.
Fermat state that from $n = 3$ it is true that

$$C^n \neq A^n + B^n$$

Keep the case $n = 3$ as example, and rewrite it in Sums:

$$A^3 = C^3 - B^3 \tag{1}$$

$$\sum_{X=1}^{A}(3X^2 - 3X + 1) \neq \sum_{X=1}^{C}(3X^2 - 3X + 1) - \sum_{X=1}^{B}(3X^2 - 3X + 1) \tag{2}$$

Apply the direct known cut):

$$\sum_{X=1}^{A}(3X^2 - 3X + 1) = \sum_{X=B+1}^{C}(3X^2 - 3X + 1) \tag{3}$$

Check each member on both side:

Maruelli's All Primes Interceptor :

Be:

$$z = \frac{n!}{n^2} * \delta_m$$

If n Is Not a Prime then $z \in \mathbb{N}$ (as the most famous Riemann Zeta gives negative even)

If n IS a Prime then $z \in \mathbb{Q} - \mathbb{N}r$ (as the most famous Riemann Zeta gives non trivial zeros)

where δ_m is the Correction Factor is defined to be:

$\delta_m = 1$ elsewhere except in:

$n = 1$ where $\delta_m = 2/3$ and in:

$n = 4$ where $\delta_m = 2/3$

One example of a suitable, still if not super elegant, δ_m factor was given to me by Massimo Di Paola :

$$\delta_m = 1 + (2/3 - 1)*\rfloor(1/(1 + |(n-1)|)) + (2/3 - 1)\lfloor*\rfloor(1/(1 + |(n-4))|)\lfloor$$

that in xls can be written as:

$$\delta_m = 1 + (2/3 - 1) * INT[1/(1 + ABS(B2 - 1)))] + (2/3 - 1) * INT[1/(1 + ABS(B2 - 4)))]$$

Riemann Hypo Proof

If you believe in the Transfinite induction theorem, than looking to my z you've seen that each trivial RH zero can be connected to an Integer value of my z (it is a non prime number), while each NON trivial RH zero can be connected to a Rational value of my z

So n^2 for my z works as *Selector* for primes, and it show, by Transfinite Induction, that the behavior of the Real Part of S is 1/2 works as a selector too, than there cannot be zero out of there.

In terms of my Two Hand Clock (that as shown can works also with Complex numbers), Real Part of S is 1/2 behave as a 12 onto a classic clock, for any Prime Number.

The long chain lead to this result comes from the observation of my z onto numbers once we use it to calculate the Number of Primes between 0 and P, or given a prime π_x find the next one is π_{x+1}.

2- How to discover the position of any primes in the primes list (what follows are not the only known methods)

With this simple trick you can understand the position of the primes “n” in the primes table or how many primes there where before the integer “n”:

The official formula is:

$$\pi(x) = \sum_{5 \le n \le x} n \left\{ \frac{(n-2)!}{n} \right\} + 2$$

Where what [X] into the braces is the non integer part of X, forced as 1

A more simple to understand method is:

- Force to 0 the integer part of Rm

- force at 1 the non integer part of the Rm value
So in case “n” is a prime it count 1, or 0 in case of non prime, so the sum from 1 to n will return exactly the number of the primes.

Since the method start from 5 we have to add 2 to remember of:

2= prime and 3= prime, missed starting from 5:

$$Pi(x) = \sum_{n=5}^{P} \left[\left\lfloor \frac{n!}{n^2} \right\rfloor - \left\lfloor \frac{n!}{n^2} \right\rfloor + \frac{1}{3} \right] + 2$$

That works as follow: (and where 1/3 pull decimal to 1 in case n is the prime 2)

$$\pi(x) = \underbrace{\sum_{n=5}^{P} \left(\underbrace{\mathrm{Int}\left\{ \left[\underbrace{\left(\frac{(n-1)!}{n}\right) - \mathrm{Int}\left(\frac{(n-1)!}{n}\right)}_{\substack{\text{This return: 0,XXX if n is a prime} \\ \text{0 if n is a non prime}}} \right] + 1/3 \right\}}_{\substack{\text{This return: 1 if n is a prime} \\ \text{0 if n is a non prime}}} \right)}_{\text{This return the number of primes between 0 and P}} + 2$$

3- How to find the next prime:

With the similar method it's possible to answer at the question:

If, for example be Pi(a) = 31 is a known prime, witch is the next prime ?

The process is the same:

- calculate the position "i" of the known "Pi" with the method (2) :

so $Pos(31) = i$ than $Pos(Pi + 1) = (i + 1)$

- than knowing that the new position (i+1) will be "easy" to - calculate the relative prime

One of the possible the tricks is:

- Knowing that P * 0 = 0 find a way to force at zero any number that has a position different from (a+1)

so first step is to calculate:

$$X \cdot \mathrm{Int}\left(\frac{\sum_{n=5}^{P} \left(\mathrm{Int}\left\{ \left[\left(\frac{(n-1)!}{n} \right) - \mathrm{Int}\left(\frac{(n-1)!}{n} \right) \right] + (1/3) \right\} \right) + 2}{i+1} \right)$$

X = unknown position of the prime $\pi(x)$

i = position of the known Pi prime number

(c) Stefano Maruelli www.maruelli.com/primes.htm

This give as result:

- 0 if n < P(i+1) - 1 if n = P(i+1)

- K if n = P(i+K)

So we have to find a tricks that gives 0 or 1 still if n = P(i+K) and avoid the indeterminate form 0/0.

For example we know that b! = 1 still if b = 0 so: int(b/b!) avoid the form 0/0

And return 1 if b=1 since if b =1 also b! = 1! = 1. So we use the:

$$\mathrm{Int}\left\{\frac{X\cdot\mathrm{Int}\left(\dfrac{\sum_{5}^{P}\left(\mathrm{Int}\left\{\left[\left(\frac{(n-1)!}{n}\right)-\mathrm{Int}\left(\frac{(n-1)!}{n}\right)\right]+(1/3)\right\}\right)+2}{i+1}\right)}{\left[X\cdot\mathrm{Int}\left(\dfrac{\sum_{5}^{P}\left(\mathrm{Int}\left\{\left[\left(\frac{(n-1)!}{n}\right)-\mathrm{Int}\left(\frac{(n-1)!}{n}\right)\right]+(1/3)\right\}\right)+2}{i+1}\right)\right]}\right\}$$

This give as result:

- 0 if n <> P(i+1)

- 1 if n = P(i+1)

So to make it working itself we can put this trick into a Sum that works from known limits where Pi(i+1) will be for sure present.

For example lower limit is: P(i)+1 and upper is: 2* Pi (as already proven see wikipedia)

Of course the tricks works with the upper limit till infinite, but has no sense.

So the "final trick" to have the P(i+1) knowing Pi is:

$$\pi(i+1)=\sum_{n=5}^{2*Pi} X\cdot\mathrm{Int}\left\{\frac{\mathrm{Int}\left(\dfrac{\sum_{5}^{P}\left(\mathrm{Int}\left\{\left[\left(\frac{(n-1)!}{n}\right)-\mathrm{Int}\left(\frac{(n-1)!}{n}\right)\right]+1/3\right\}\right)+2}{\sum_{5}^{P}\left(\mathrm{Int}\left\{\left[\left(\frac{(n-1)!}{n}\right)-\mathrm{Int}\left(\frac{(n-1)!}{n}\right)\right]+1/3\right\}\right)+3}\right)}{\left[\mathrm{Int}\left(\dfrac{\sum_{5}^{P}\left(\mathrm{Int}\left\{\left[\left(\frac{(n-1)!}{n}\right)-\mathrm{Int}\left(\frac{(n-1)!}{n}\right)\right]+1/3\right\}\right)+2}{i+1}\right)\right]!}\right\}$$

All that works as a very slow computer program, so has no sense for make a real calculation, but can give you an idea of what make Primes soo hard to be discovered.

So is necessary to "process" all the numbers from 5 to X each time, and for several times...)

But we cannot say longer that "is impossible to find a formula to calculate the next prime".

And finally we hazard to say that seems now more probable that there will not be an absolutely easy function that feet all primes.

Of course there are other more faster algorithm to find primes (for example Eartostene method) but, in my opinion, they will not give a “sense” of how prime are made as Wilkinson theorem (and what follow from it).

There is non official formula discovered in 1964 that involves sin(x) and Integer operator too.

4- My final concerning on: I try to go over saying that is more clear now why complex

numbers can well fit the primes calculation:

complex numbers, as primes, has 2 non connected “parts”:

the real one and the complex,

as prime can be connected to a number $z = \frac{n!}{n^2}$ that has

an integer part, that is common to a non primes numbers,

and a non integer part that is unique and non present in non primes numbers.

You can find animated Gif, upgrade and other info at my webpage:

http://shoppc.maruelli.com/prime-study.htm

References:

There is no reference for what I invented, for all the Rest is standard Math so you can find reference elsewhere on the web / books /e-books.

The most related paper I've found is:

Title: Using the Finite Difference Calculus to Sum Powers of Integers

Author: Lee Zia Reviewed work

Source: The College Mathematics Journal, Vol. 22, No. 4 (Sep., 1991), pp. 294-300

Published by: Mathematical Association of America Stable

http://www.jstor.org/stable/2686229

The document is interesting, require a little higher Math skill, and introduce same concept of "finite differences" you'll find here, but in a more general way. Unfortunately the article stops when the thinks become interesting, so I hope to give to reader some more detail and info on the telescopic sum properties.
Interesting papers:

- Set Theory: Counting the Uncountable Waffle - Mathcamp 2012

http://www.math.harvard.edu/ waffle/settheory.pdf